A Measure of Added Value in Groups

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The intuitive notion of *added value* in groups represents a fundamental property of biological, physical, and economic systems: how the interaction or cooperation of multiple entities, substances, or other agents can produce synergistic effects. However, despite the ubiquity of group formation, a well-founded measure of added value has remained elusive. Here, we propose such a measure inspired by the *Shapley value*—a fundamental solution concept from Cooperative Game Theory. To this end, we start by developing a solution concept that measures the *average impact* of each player in a coalitional game and show how this measure uniquely satisfies a set of intuitive properties. Then, building upon our solution concept, we propose a measure of added value that not only analyzes the interactions of players *inside* their group, but also *outside* it, thereby reflecting otherwise-hidden information about how these individuals typically perform in various groups of the population.

CCS Concepts: • **Mathematics of computing** → *Permutations and combinations*; *Combinatorial algorithms*;

Additional Key Words and Phrases: Synergy, Harsanyi dividends, interaction index, Shapley value

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1 INTRODUCTION

Groups produce added value in numerous diverse settings, ranging from the organic to the inorganic and from the micro level to the macro level. For instance, the genes in our genome collectively produce seemingly coordinated outcomes (Gregory 2005). Similarly, various elements can be grouped together to form wholly new and completely different compounds (Horkov et al. 2014). The same holds for ant colonies (Wheeler 1911), flocks of migrating birds (Nathan and Barbosa 2008), and communities of mammals (Steinmetz et al. 2008), all of which yield outcomes that are

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far beyond the reach of individual members. Added value also emerges at the interface of living organisms and objects. For instance, early humans combined their skills with objects to achieve things that otherwise would be impossible, and this eventually facilitated an unprecedented development of civilization (Corning 2005).

To illustrate some of the subtleties involved in developing a measure of added value, suppose we have a number of employees in a company and a number of tasks that require the formation of different groups throughout the year. Suppose further that the Human Resources (HR) department of the company is asked to assess the added value in different groups, e.g., to identify the groups that bring out the best in their members.[1] What would be a principled way to go about this? Suppose that groups $\{1, 2, 3\}$ and $\{4, 5, 6\}$ have produced an equal outcome of, say, USD 50,000 each. A naive measure may consider the added value to be equal in both groups, since they produce the same outcome. However, suppose that:

— Each of the employees in $\{1, 2, 3\}$ has good leadership skills. As such, he or she usually makes a substantial impact on the group performance. However, when these employees are placed in the same group with no one else, they argue constantly, each trying to prove more worthy of leadership than the others, which ultimately undermines the overall performance of the group. In this case, the HR department may consider an outcome of USD 50,000 to be disappointing, since this outcome is below expectation from such usually highly productive employees.
— However, suppose each of the employees in $\{4, 5, 6\}$ is introverted, which usually inhibits him or her from making any substantial contribution to his or her group. Suppose further that the three employees happen to be close friends. As such, when working together in a group with no one else, they overcome their teamwork problems, leading to an outcome of USD 50,000—an outcome that indicates an improvement in their overall performance *bearing in mind the small contribution that each of them typically makes to a group*. The HR department may then conclude that putting those employees together in a group improves their performance and yields added value.

So, even though $\{1, 2, 3\}$ and $\{4, 5, 6\}$ produce the same outcome, the added value seems to be negative in the first group and positive in the second. More generally, a group's outcome does not necessarily reflect the added value therein, since that outcome contains no information about how the members typically perform in other groups.

An alternative way to measure the added value would be to compare the outcome of the group with the *expected* outcome of each member. From this perspective, the added value seems to reflect the degree to which the group outcome *deviates from expectation*; any group that exceeds those expectations is deemed to yield positive added value, and any group that falls short of them is deemed to yield negative added value. This is, for example, how an outcome of USD 50,000 was considered *disappointing* when produced by the group of leaders, while that same outcome was considered *exceptional* when produced by the group of introverts. The main question is then: *How do we measure the expected outcome of each member?*

One way to answer this question is by considering the *expected* outcome of each member to be equal to the outcome produced when that member works *alone*. This leads to a measure with which the added value is computed as *the difference between the outcome of the group and the outcomes attainable when each member works separately* (see, e.g., Larson 2010). Although this measure is rather intuitive, it fails to account for any group-related attributes of the individuals involved. For

[1]This is a natural question to ask, e.g., given the evidence found by Woolley et al. (2010) of "collective intelligence" in human groups.

example, coming back to our company scenario, suppose that a certain employee is very skilled but antisocial; this person excels when working alone, but consistently underperforms when working in any group, *regardless of the identities of his or her co-workers*. To obtain a measure that takes such information into account, one clearly needs to look beyond just the outcome that each member achieves when working alone. For example, in our company scenario, the outcome of the group of introverted employees was deemed satisfactory *bearing in mind the social nature of those employees*, which cannot be inferred from just the group's outcome, nor from the outcome that each member achieves when working alone.

Other measures of synergy include the *Harsanyi dividends* (Harsanyi 1958), the *interaction index* (Grabisch and Roubens 1999), and the measure whereby synergy is taken as the difference between the outcome of the group and the outcome of an *optimal partition* of that group (Conitzer and Sandholm 2006). These alternative synergy measures will be discussed thoroughly in Section 7, but for now let us focus on the group $\{1, 2, 3\}$ in a four-player setting to illustrate how the Harsanyi dividends and interaction index work. The first resembles the way in which the inclusion-exclusion principle works. More specifically, according to the Harsanyi dividends, the synergy in $\{1, 2, 3\}$ is computed as follows:

$$\begin{aligned}\text{Synergy}(\{1,2,3\}) = {} & \text{outcome}(\{1,2,3\}) - \text{outcome}(\{1,2\}) - \text{outcome}(\{1,3\}) - \text{outcome}(\{2,3\}) \\ & + \text{outcome}(\{1\}) + \text{outcome}(\{2\}) + \text{outcome}(\{3\}).\end{aligned}$$

In contrast, the interaction index analyzes how the different subsets (including the empty set) perform when combined with non-members. Since we are focusing on $\{1, 2, 3\}$ in a four-player setting, the only non-member is 4. As such, according to the interaction index, the synergy in $\{1, 2, 3\}$ is computed as follows:

$$\begin{aligned}\text{Synergy}(\{1,2,3\}) = {} & \text{outcome}(\{1,2,3,4\}) - \text{outcome}(\{1,2,4\}) - \text{outcome}(\{1,3,4\}) - \text{outcome}(\{2,3,4\}) \\ & + \text{outcome}(\{1,4\}) + \text{outcome}(\{2,4\}) + \text{outcome}(\{3,4\}) - \text{outcome}(\{4\}).\end{aligned}$$

Importantly, none of the alternative synergy measures reflects the intuitive notion of synergy that we have discussed in our company scenario. To demonstrate this point, let us consider the sample scenario illustrated in Figure 1, which specifies the revenue generated by each employee in every group that he/she may belong to. Here, employees 1 and 2 generate significantly more revenue whenever employee 4 is part of the group. In contrast, employee 3 is antisocial, and his/her revenue decreases with the team size, unlike employee 4. The outcome of every group is then computed as the sum of the revenues of its members, e.g., the outcome of $\{2, 3\}$ is $30 + 40 = 70$ and the outcome of $\{1, 2, 3\}$ is $40 + 40 + 30 = 110$. Note that, in this stylized example, the contribution of each member is given explicitly, e.g., the contribution of 1 to $\{1, 2, 3\}$ is 40. This entire example is discussed more thoroughly in Section 7, but for now let us focus on some interesting cases. Starting with the group $\{1, 2\}$, one can argue that both employees underperform, since each of them produces a smaller revenue than his/her average (or expected) revenue. Despite this fact, as can be seen in Table 1, none of the aforementioned synergy measures assigns a negative added value to $\{1, 2\}$. Similarly, the formation of $\{1, 2, 3\}$ causes each employee therein to perform below his/her average, and yet none of the synergy measures assigns a negative added value to this group. Finally, note that the employee 1 performs the worst when working alone, and yet no synergy measure is able to detect this fact (see Table 1). Thus, there is a need to develop a measure that assigns negative added value to such groups. This becomes even more challenging in scenarios where, unlike our stylized example, the outcome of a group is not broken down into the revenues generated by each member and, as such, it is challenging to determine the contribution of each member to start with.

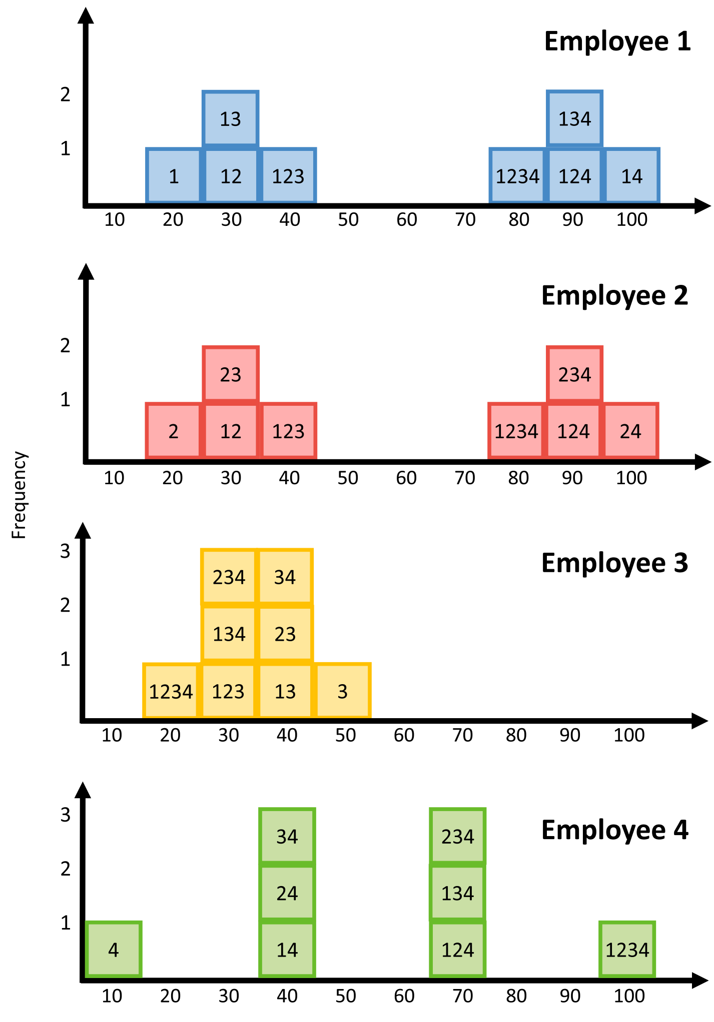

Fig. 1. A stylized scenario, where every group member generates a revenue, and the outcome of the group equals the sum of the revenues of its members. In particular, the subfigures specify the revenue generated by employees 1, 2, 3, and 4 when working in different groups, e.g., the revenue generated by employee 1 equals 20 when working alone, 30 when working in either $\{1, 2\}$ or $\{1, 3\}$, and so on. (This is indeed a histogram, since the height of each bar represents the frequency of any given revenue, e.g., there is a single group in which the revenue of 1 equals 20, and two groups in which the revenue of 1 is 30, etc.) The outcome of each group is the sum of the revenues of its members, e.g., the outcome of $\{2, 3\}$ is $30 + 40 = 70$ and the outcome of $\{1, 2, 3\}$ is $40 + 40 + 30 = 110$. This example is discussed more thoroughly in Section 7.

Against this background, we set out to develop a measure that interprets the added value in a group as the difference between the outcome of that group and the *expected contribution of each member*. Our measure is able to overcome the limitations of the aforementioned measures, e.g., given our stylized scenario from Figure 1 and Table 1, our measure assigns negative added value to each of the discussed groups, i.e., $\{1\}$, $\{1, 2\}$, and $\{1, 2, 3\}$, unlike the other measures. We do so in an informed way, inspired by solutions concepts from *Cooperative Game Theory*—a branch of

Table 1. Given the Scenario Illustrated in Figure 1, the Table Specifies the Added Value in the Groups $\{1\}$, $\{1,2\}$ and $\{1,2,3\}$ According to Different Measures of Synergy

Group	Group outcome	Outcome minus the sum of the individuals' outcomes	Harsanyi dividends	Interaction index	Outcome minus the outcome of an optimal partition
$\{1\}$	20	0	20	65	20
$\{1,2\}$	60	20	20	0	20
$\{1,2,3\}$	110	20	0	0	0

microeconomic theory that studies the formation of coalitions and how the rewards from cooperation should be divided among participants. More specifically, our inspiration comes from the *Shapley value*—the canonical solution concept in the literature for assessing the contribution of each group member in a fair way. In this context, the entire setting is called a *"game,"* every individual (e.g., every employee in the company) is called a *"player,"* every possible group (i.e., every possible subset of players) is called a *"coalition,"* and the outcome of a coalition is called the *"value"* of that coalition. Roughly speaking, the Shapley value reflects the impact that each player makes to the value of the *grand coalition*—the coalition consisting of all players in the game—taking into consideration the values of all possible coalitions. Inspired by the Shapley value, we formulate a *measure of average impact*, designed to reflect the impact that each player makes to the value of an *arbitrary* coalition. In a nutshell, this measure is the *average Shapley value* taken over all subgames of the game at hand (see Section 3 for more details). We identify a number of "axioms" that define this measure of average impact, i.e., a number of properties that cannot all be satisfied by any other measure. With such an axiomatic system, one would be able to make an informed decision about whether or not to use the measure. That is, in settings where the axioms happen to be meaningful and desirable, the measure would be a reasonable choice, whereas in settings where the axioms seem to be unattractive, other measures should be considered instead. We prove in Section 3 that the following properties axiomatize our measure of average impact[2]:

(1) *Symmetric-Impact*: this axiom requires that any two players that are *symmetric* players (i.e., can be substituted with one another in any group without affecting the outcome of that group) have the same average impact;
(2) *Marginal-Impact*: this axiom requires that the average impact of a player is measured based on the absolute, rather than the relative, impacts that he or she makes in different coalitions;
(3) *Average-Value*: this axiom requires that if every player in the entire game makes his or her average impact, we obtain the average coalition value. To put it differently, the average impacts lead to an average outcome.

After that, in Section 4, we use our measure of average impact to develop a new measure of added value, which quantifies the *deviation from expectation* by comparing the coalition's value against the *average impacts* of its members. Here, we identify two sets of axioms, each of which is uniquely satisfied by our measure of added value. The first consists of the following five axioms:

(1) *Symmetric-Value*—replacing a player with another symmetric player does not alter the added value in the coalition;

[2] A more formal definition of each axiom can be found in Section 3.

(2) *Dummy-Value*—by subtracting the added values from the coalition values in the entire game, we end up with a new, additive game in which every player produces exactly the same outcome[3] regardless of the coalition he or she belongs to;
(3) *Additive-Value*—this axiom implies that scaling the coalition values does not alter the relative differences between the added values in different coalitions;
(4) *Null-Value*—if a player has no impact on any coalition's value, then that player has no impact on any coalition's added value (because nothing is produced by, nor expected from, such a player, meaning that his or her membership does not alter the degree to which a coalition *deviates from expectation*);
(5) *Normalized-Value*—the sum of all added values is zero. This is related to our interpretation of added value as a form of *deviation from expectation*: some coalitions may yield positive added value (i.e., they exceed our expectations) while others may yield negative added value (i.e., they fall short of our expectations), but on average the coalitions meet our expectations (i.e., the average of all added values is zero), otherwise our expectations would be wrong to start with.

In the second axiomatization of our measure of added value (Section 5), we show that it is possible to replace the *Symmetric-Value* and *Additive-Value* axioms with what we call the *Marginal-Value* axiom. Roughly speaking, this axiom requires that the added value in a coalition is not affected by a change in the performance of non-members.

In Section 6, we show how our measure of added value can be computed in polynomial time under a concise, widely used representation of the different coalition values, namely the *Marginal-Contribution network* (or *MC-net*) representation (Ieong and Shoham 2006).

Finally, in Section 7, we discuss how our measure of added value stands in relation to other measures of synergy in the literature and highlight the main differences through a number of examples. A table summarizing the main notation can be found in Appendix A.1.

2 PRELIMINARIES

Throughout the article, we will use standard concepts and notation from cooperative game theory. In particular, the entire setting will be referred to as a *characteristic-function game*, defined by a pair (N, v) such that $N = \{1, \ldots, n\}$ is the set of individuals or "*players*" and $v : 2^N \rightarrow \mathbb{R}$ is the *characteristic function*. This function assigns to every possible subset or *"coalition"* $C \subseteq N$ a real value, $v(C)$—called *the value of C*—which can be interpreted as the utility attainable by that coalition. For instance, in a game that represents our company scenario, every player represents an employee, while the characteristic function returns the outcome generated by any given group. As is common in the literature, we will assume that $v(\emptyset) = 0$. We will denote by $\mathscr{G}(N)$ the set of all possible games defined over the set of players N. More formally, $\mathscr{G}(N) = \{(N, v) | v : 2^N \rightarrow \mathbb{R}\}$.

The following standard definitions will be used repeatedly throughout the article:

Definition 2.1 (Sub-game). Given a game, (N, v), and a coalition, $S \subseteq N$, the "*sub-game*" (S, v) is the game in which the set of players is S, and the value of every coalition, $C \subseteq S$, is $v(C)$.

Definition 2.2 (Symmetric Players). Let (N, v) be a game. Players $i, j \in N$ are *symmetric* iff: $v(C \cup \{i\}) = v(C \cup \{j\})$ for every $C \subseteq N \setminus \{i, j\}$.

From a strategic perspective, any two symmetric players are identical in the game; they only differ in their names.

[3]The term "dummy" is inspired by the term "dummy player," which is standard in coalitional game theory to refer to a player whose addition to a coalition does not increase the value of that coalition (Chalkiadakis et al. 2011).

Definition 2.3 (Null Player). Let (N, v) be a game. A player $i \in N$ is a *null player* iff $v(C \cup \{i\}) = v(C)$ for all $C \subseteq N$.[4]

That is, whenever a null player joins a coalition, he or she has no impact on that coalition's value.

Definition 2.4 (Dummy Player). Let (N, v) be a game. A player $i \in N$ is a *dummy player* iff $v(C \cup \{i\}) = v(C) + v(\{i\})$ for all $C \subseteq N \setminus \{i\}$.

Intuitively, a dummy player is one whose performance is never affected by his or her teammates.

Definition 2.5 (Sum of Two Games). The *sum of two games*, (N, v) and (N, w), is a game denoted by $(N, v + w)$, where $(v + w)(C) = v(C) + w(C)$ for every coalition $C \subseteq N$.

Similarly, subtracting (N, v) from (N, w) results in a game denoted by $(N, v - w)$, where $(v - w)(C) = v(C) - w(C)$ for every coalition $C \subseteq N$.

Definition 2.6 (Marginal Contribution). Let (N, v) be a game. The *marginal contribution* of a player $i \in N$ to a coalition $C \subseteq N$ is:

$$MC_i^C(N, v) = v(C \cup \{i\}) - v(C).$$

Note that $MC_i^C(N, v) = 0$ for all $i \in N$ and all $C \subseteq N : i \in C$. The next definition shows how the Shapley value is based on the notion of marginal contribution.

Definition 2.7 (Shapley Value). Let (N, v) be a game, and let $i \in N$. The *Shapley value* of i is denoted by $\phi_i(N, v)$ and given by:

$$\phi_i(N, v) = \sum_{C \subseteq N \setminus \{i\}} \frac{|C|!\,(|N| - |C| - 1)!}{|N|!} \Big(v(C \cup \{i\}) - v(C)\Big). \tag{1}$$

Assuming that the largest coalition, i.e., N, will be formed, and that we want to divide $v(N)$ among the players, Shapley showed that $\phi_1(N, v), \ldots, \phi_n(N, v)$ is the only possible division of $v(N)$ that satisfies all of the following properties (Shapley 1953):

Symmetry: For every game $(N, v) \in \mathcal{G}(N)$ and every pair of symmetric players i and j in the game: $\phi_i(N, v) = \phi_j(N, v)$;
Null Player: For every game $(N, v) \in \mathcal{G}(N)$, and every null player i in the game: $\phi_i(N, v) = 0$;
Efficiency: For every game $(N, v) \in \mathcal{G}(N)$, we have: $\sum_{i \in N} \phi_i(N, v) = v(N)$;
Additivity: For every pair of games, (N, v) and (N, w), and every player $i \in N$, we have: $\phi_i(N, v + w) = \phi_i(N, v) + \phi_i(N, w)$.

The above four properties are commonly known as the Shapley axioms. Finally, note that Appendix A.1 summarizes the main notation used throughout the article.

3 A MEASURE OF AVERAGE IMPACT

As mentioned earlier in the introduction, we interpret the added value as a form of *deviation from expectation*. Thus, to measure the added value in a given coalition, we need to first quantify the *expected outcome* of that coalition. It seems reasonable for such an outcome to be the result of each member making his or her *average impact*. To better understand the rationale behind this, consider an example where the goal is to measure the added value in a group of scientists who wrote an

[4] If a coalition C contains player i, then the equality holds trivially, because then $C \cup \{i\} = C$.

article together. For the sake of simplicity, suppose that the worth of an article is measured based on its number of citations. Arguably, this number alone does not adequately reflect the added value among the authors, because it does not contain any information about the number of citations that each of those authors usually brings to a paper. For instance, suppose that the paper has 30 citations. Then, to measure the added value among the authors, it makes sense to first consider the average number of citations that each of them has *with other collaborators*. If, for example, each of those authors usually has far more than 30 citations per article, then this suggests that those authors are underperforming in this particular group. Conversely, if none of them ever wrote any other article whose number of citations reached 30, then this suggests that those authors perform better than usual when working together as a group, i.e., there is a positive added value between them. Again, the main question becomes: *how do we quantify the average impact of each scientist?*

Against this background, our goal in this section is to develop a measure that quantifies the *average impact* of each player in a coalitional game, i.e., the *average amount of utility that the player contributes towards the value of a coalition*. To put it differently, we set out to answer the following question:

> *Judging from the values of the different coalitions in a game, how can we quantify the average impact that a particular player makes?*

This is particularly relevant when all what we know for a fact are the values themselves, and not the exact factors that may have influenced those values. Again, consider our company scenario, where the impact that an employee has on a group may depend on various factors, such as the employee's reputation, the past experiences that he or she had with other members, and so on. Some of these factors are very hard (if not impossible) to observe or quantify. However, the group outcomes are observable and comparable. It is exactly such cases that we focus on in our study. In such cases, it would be desirable to have the ability to measure the average impact of a player, depending solely on the values of different coalitions.

To this end, let us first define a *measure of average impact*, θ, as a function that assigns to every player $i \in N$ a real value representing the impact that i makes on average *on the value of any given coalition*. As such, if we denote the set of all such measures by Θ, and denote by $\mathscr{G}(N)$ the set of all possible games defined over a set of players, N, then every $\theta \in \Theta$ is a real-valued function defined for every game $(N, v) \in \mathscr{G}(N)$ and every player i in that game. More formally, θ is a real-valued function on the set:

$$\bigcup_{(N,v)\in\mathscr{G}(N)} \Big\{(N, v, i) : i \in N\Big\}.$$

For notational convenience, we will write $\theta_v^N(i)$ instead of $\theta((N, v, i))$. Importantly, $\theta_v^N(i)$ is the average impact of i *on any given coalition that contains* i. While there are no restrictions on how $\theta_v^N(i)$ is defined or computed, it would be helpful to identify a number of desiderata to guide the development of such a measure of average impact. The following are seemingly reasonable properties to have in a measure of average impact $\theta \in \Theta$ (a discussion of those properties will follow):

P_1. (**Symmetric-Impact**): A measure $\theta \in \Theta$ satisfies P_1 if, for every game $(N, v) \in \mathscr{G}(N)$, and every pair of symmetric players i and j in that game, we have:

$$\theta_v^N(i) = \theta_v^N(j);$$

P_2. (**Average-Value**): A measure $\theta \in \Theta$ satisfies P_2 if, for every game $(N, v) \in \mathcal{G}(N)$, we have:

$$\sum_{i \in N} \theta_v^N(i) = 2^{1-|N|} \sum_{C \subseteq N} v(C);$$

P_3. (**Marginal-Impact**): A measure $\theta \in \Theta$ satisfies P_3 if, for every pair of games (N, v) and (N, w) with the same set of players, and for every player $i \in N$, if

$$\forall C \subseteq N \setminus \{i\}, \quad MC_i^C(N, v) = MC_i^C(N, w), \tag{2}$$

then the following holds:

$$\theta_v^N(i) = \theta_w^N(i).$$

Let us now comment on the above properties. Starting with P_1, this property states that symmetric players have identical average impacts, which seems rather intuitive, since such players can be substituted with one another in any given coalition without changing the value of that coalition, i.e., they are identical except for having different names. As such, it would be strange for a measure of average impact to distinguish between them.

As for P_2, it states that the average outcome of the game is obtained when every player in that game makes his or her average impact, which again seems intuitive.[5]

As for P_3, this property states that if the performance (in terms of *marginal contributions*; see Definition 2.6) of a certain player remained unchanged, then the average impact of that player should also remain unchanged, regardless of whether the performance of some other player(s) in the game has changed. This implies that the measure reflects the absolute, rather than relative, impact of the player. For example, the average impact of an employee could be, say, USD 10,000 (an absolute value), rather than being, say, 10% of the group outcome (which is a relative value).

Inspired by the above properties, we propose a measure of average impact, called the *Average-Impact index*, which is defined as follows[6]:

Definition 3.1 (Average-Impact Index). The *Average-Impact index* is the measure of average impact, $\alpha \in \Theta$, defined for every game $(N, v) \in \mathcal{G}(N)$ and every player $i \in N$ as follows:

$$\alpha_v^N(i) = \overline{\phi}_i(N, v), \tag{3}$$

where $\overline{\phi}_i(N, v)$ is the **average Shapley value** of player i taken over all *sub-games* $(S, v) : S \subseteq N, i \in S$ (see Definition 2.1). More formally,

$$\overline{\phi}_i(N, v) = 2^{1-|N|} \sum_{S \subseteq N : i \in S} \phi_i(S, v). \tag{4}$$

The following theorem implies that properties P_1, P_2, and P_3 axiomatize the Average-Impact index.

Theorem 3.2. *The Average-Impact index is the only measure in Θ that satisfies properties P_1, P_2, and P_3.*

Proof. We begin by proving that the Average-Impact index satisfies the three properties listed in the statement of the theorem.

[5]This is because $\theta_v^N(i)$ is the average impact of i *on all coalitions that contain* i, while $\frac{1}{2}\theta_v^N(i)$ is the average impact of i *on all coalitions in the game*. This is why we have $2^{1-|N|}$ instead of $2^{-|N|}$ in P_2.

[6]The main reason behind calling our measure the "*Average Impact index*" rather than the "*Average Shapley value*" is due to the fact that the "*Average Shapley value*" does not specify how the average is computed. For instance, it could be interpreted as the average taken *over all players*, rather than *over all subgames*, e.g., as in Moretti (2009), or it could be interpreted as a *weighted* average. Due to those different interpretations, it would be imprecise to write that we "*axiomatize the Average Shapley value.*" Instead, we write that we "*axiomatize the Average Impact index*," which is defined formally in Definition 3.1.

Claim 1. *The Average-Impact index satisfies properties P_1, P_2, and P_3.*

Proof. First, to prove that α satisfies Property P_1, it suffices to prove that for every pair of symmetric players $i, j \in N$, the following holds:

$$\overline{\phi}_i(N, v) = \overline{\phi}_j(N, v). \tag{5}$$

To this end, observe that:

— For every $C \subseteq N : \{i, j\} \subseteq C$, the players i and j are symmetric in the sub-game (C, v). This, as well as Shapley's *Symmetry* axiom from Section 2, implies that $\phi_i(C, v) = \phi_j(C, v)$.

— For every $C \subseteq N \setminus \{i, j\}$, let us analyze the sub-games $(C \cup \{i\}, v)$ and $(C \cup \{j\}, v)$. We know from Equation (1) that the following holds:

$$\phi_i(C \cup \{i\}, v) = \sum_{S \subseteq C} \frac{|S|!\ ((|C| + 1) - |S| - 1)!}{(|C| + 1)!} \Big(v(S \cup \{i\}) - v(S)\Big), \tag{6}$$

$$\phi_j(C \cup \{j\}, v) = \sum_{S \subseteq C} \frac{|S|!\ ((|C| + 1) - |S| - 1)!}{(|C| + 1)!} \Big(v(S \cup \{j\}) - v(S)\Big). \tag{7}$$

Now, since i and j are symmetric, and since every $S \subseteq C$ does not contain i nor j, then from the definition of symmetry (i.e., Definition 2.2), we know that: $v(S \cup \{i\}) = v(S \cup \{j\})$. This, as well as Equations (6) and (7), imply that $\phi_i(C \cup \{i\}, v) = \phi_j(C \cup \{j\}, v)$.

The above two cases imply that Equation (5) holds.

Moving on to P_2, we have:

$$\begin{aligned}
\frac{1}{2} \sum_{i \in N} \overline{\phi}_i(N, v) &= 2^{-|N|} \sum_{i \in N} \sum_{C \subseteq N : i \in C} \phi_i(C, v) \quad \text{(following Equation (4))} \\
&= 2^{-|N|} \sum_{C \subseteq N} \sum_{i \in C} \phi_i(C, v) \\
&= 2^{-|N|} \sum_{C \subseteq N} v(C). \quad \text{(as } \phi \text{ satisfies Shapley's } \textit{Efficiency} \text{ axiom)}
\end{aligned}$$

Consequently, α satisfies P_2.

Finally, to prove that α satisfies P_3, it suffices to point out that, for every $i \in N$, the average Shapley value, $\overline{\phi}_i(N, v)$, is by definition a weighted average of the marginal contributions of i to all the coalitions in the game (N, v) (see Equations (1) and (4)). □

Having proved the correctness of Claim 1, it remains to prove the uniqueness statement in Theorem 3.2. For this, we need to introduce two additional definitions:

Definition 3.3 ($C(N, v)$). For every game $(N, v) \in \mathcal{G}(N)$, the set $C(N, v) \subseteq 2^N$ is the set consisting of every coalition in (N, v) that has at least one subset whose value is non-zero. Formally,

$$C(N, v) = \{C \subseteq N : \exists C' \subseteq C, v(C') \neq 0\}. \tag{8}$$

Definition 3.4 ((N, v_S)). For every set of players $N = \{1, 2, \ldots, n\}$, and every coalition, $S \subseteq N$, and every characteristic function $v : 2^N \to \mathbb{R}$, the game (N, v_S) is defined as follows:

$$v_S(C) = v(C) - v(C \cap S), \quad \forall C \subseteq N. \tag{9}$$

With these definitions in place, we are now ready to prove the uniqueness statement in Theorem 3.2. Assuming that $\delta \in \Theta$ is an average-impact measure satisfying P_1, P_2, and P_3, we will prove that $\delta = \alpha$. The proof will be an inductive one over $|C(N,v)|$. More specifically, In **Step 1** below, we will prove that $\delta = \alpha$ for every game $(N,v) \in \mathcal{G}(N)$ such that $|C(N,v)| = 0$, i.e., we will prove:

$$\delta = \alpha, \ \ \forall (N,v) \in \mathcal{G}(N) : |C(N,v)| = 0. \tag{10}$$

After that, in **Step 2** below, we will prove that if we have $\delta = \alpha$ for every game $(N,v) \in \mathcal{G}(N)$ such that $|C(N,v)|$ is smaller than some size $s \in \mathbb{N} \setminus \{0\}$, then we also have $\delta = \alpha$ for every game $(N,v) \in \mathcal{G}(N)$ such that $|C(N,v)|$ equals s, i.e., we will prove:

$$\delta = \alpha, \ \forall (N,v) \in \mathcal{G}(N) : |C(N,v)| < s \quad \Rightarrow \quad \delta = \alpha, \ \forall (N,v) \in \mathcal{G}(N) : |C(N,v)| = s. \tag{11}$$

Step 1: Proving that Equation (10) holds:
Definition 3.3 implies that, for every set of players $N = \{1, 2, \ldots, n\}$, there exists exactly one game $(N,v) \in \mathcal{G}(N)$ for which $|C(N,v)| = 0$; this is the game in which every coalition's value is zero, i.e., it is the game (N, v^0), where $v^0(C) = 0, \forall C \subseteq N$. In this game, every pair of players are symmetric. Consequently, the following holds, as both δ and α satisfy Property P_1:

$$\delta^N_{v^0}(i) = \delta^N_{v^0}(j), \ \ \forall i,j \in N, \tag{12}$$

$$\alpha^N_{v^0}(i) = \alpha^N_{v^0}(j), \ \ \forall i,j \in N. \tag{13}$$

Furthermore, since both δ and α satisfy Property P_2, then:

$$\sum_{i \in N} \delta^N_{v^0}(i) = \sum_{i \in N} \alpha^N_{v^0}(i). \tag{14}$$

Equations (12), (13), and (14) imply that $\delta^N_{v^0} = \alpha^N_{v^0}$, meaning that Equation (10) holds, which is what we wanted to prove in Step 1.

Step 2: Proving that Equation (11) holds:
Assuming that the following holds:

$$\forall (N,v) \in \mathcal{G}(N) : |C(N,v)| < s, \ \ \forall i \in N, \ \ \delta^N_v(i) = \alpha^N_v(i), \tag{15}$$

we need to prove that the following holds for every $i \in N$ for every characteristic function, w, that satisfies $|C(N,w)| = s$:

$$\delta^N_w(i) = \alpha^N_w(i). \tag{16}$$

Importantly, for every game $(N,v) \in \mathcal{G}(N)$, and every $S \in C(N,v)$, Young proved that $C(N, v_S) \subset C(N,v)$ (see Young (1985)). This implies that[7] $|C(N,w_S)| < |C(N,w)|$, meaning that $|C(N,w_S)| < s$. Thus, based on our assumption that Equation (15) holds, we find that:

$$\forall i \in N, \ \ \delta^N_{w_S}(i) = \alpha^N_{w_S}(i). \tag{17}$$

Now, denote by $\widehat{C}$ the set formed by the intersection of all coalitions in $C(N,w)$, i.e.,

$$\widehat{C} = \bigcap_{C \in C(N,w)} C. \tag{18}$$

We will prove the correctness of Equation (16) in two steps, based on $\widehat{C}$. Specifically, in Step 2.1, we will prove that the equation holds for every $i \in N \setminus \widehat{C}$, while in Step 2.2, we will prove that it holds for every $i \in \widehat{C}$.

[7]Note that w_S is derived from w in the same way that v_S is derived from v in Definition 3.4.

Step 2.1. For every $i \in N \setminus \widehat{C}$, there exists a coalition in $\mathcal{C}(N, w)$ that does not contain i. Let S be one such coalition. Then, for every $C \subseteq N$, we have:

$$\begin{aligned} MC_i^C(N, w_S) &= w_S(C \cup \{i\}) - w_S(C) \quad \text{(based on Definition 2.6)} \\ &= w(C \cup \{i\}) - w((C \cup \{i\}) \cap S) - w(C) + w(C \cap S) \quad \text{(Definition 3.4)} \\ &= w(C \cup \{i\}) - w(C \cap S) - w(C) + w(C \cap S) \quad \text{(because } i \notin S) \\ &= w(C \cup \{i\}) - w(C) \\ &= MC_i^C(N, w). \quad \text{(based on Definition 2.6)} \end{aligned}$$

Consequently, the following two equations hold, as both δ and α satisfy Property P_3:

$$\forall i \in N \setminus \widehat{C}, \ \ \delta_{w_S}^N(i) = \delta_w^N(i), \tag{19}$$

$$\forall i \in N \setminus \widehat{C}, \ \ \alpha_{w_S}^N(i) = \alpha_w^N(i). \tag{20}$$

Equations (17), (19), and (20) imply that Equation (16) holds for every $i \in N \setminus \widehat{C}$, which is what we wanted to prove in Step 2.1.

Step 2.2. In this step, we want to prove that Equation (16) holds for every $i \in \widehat{C}$. In the case where $\widehat{C} = \emptyset$, the conclusion follows vacuously. Next, we will deal with the case where $|\widehat{C}| = 1$, and then deal with the case where $|\widehat{C}| > 1$. In both cases, we will make use of the fact that the following holds, as both δ and α satisfy Property P_2:

$$\sum_{i \in N} \delta_w^N(i) = \sum_{i \in N} \alpha_w^N(i). \tag{21}$$

First, assume that $|\widehat{C}| = 1$, and let b denote the only player in $\widehat{C}$. Since we know that Equation (16) holds for every $i \in N \setminus \{b\}$, then Equation (21) implies that $\delta_w^N(b) = \alpha_w^N(b)$, i.e., it implies that Equation (16) holds for the one player in $\widehat{C}$, which is what we wanted to show.

Now, assume that $|\widehat{C}| > 1$. Definition 3.3 and Equation (18) imply that every coalition C that does not contain $\widehat{C}$ satisfies: $w(C) = 0$. This implies that, for every pair of players, $i, j \in \widehat{C}$, and every $C \subseteq N \setminus \{i, j\}$, we have: $w(C \cup \{i\}) = w(C \cup \{j\}) = 0$. Therefore, every pair of players in $\widehat{C}$ are symmetric in the game (N, w). Consequently, the following holds, as both δ and α satisfy Property P_1:

$$\delta_w^N(i) = \delta_w^N(j), \ \ \forall i, j \in \widehat{C}, \tag{22}$$

$$\alpha_w^N(i) = \alpha_w^N(j), \ \ \forall i, j \in \widehat{C}. \tag{23}$$

Recall that we proved in Step 2.1 that Equation (16) holds for every $i \in N \setminus \widehat{C}$. Based on this, Equations (21), (22), and (23) imply that Equation (16) also holds for every $i \in \widehat{C}$, which is what we wanted to prove. This concludes Step 2.2, and so concludes the proof of Theorem 3.2. □

Having proposed $\alpha \in \Theta$ as a measure of the average impact of a player in a game $(N, v) \in \mathcal{G}(N)$, in the following section, we use α as a building block to construct our measure of added value.

4 A MEASURE OF ADDED VALUE

In this section, we develop a measure that quantifies the added value in any given coalition. Unlike a measure of average impact, which is defined for every game and *every player* in that game, a measure of added value is defined for every game and *every subset of players* in that game. More formally, it is defined as follows:

Definition 4.1 (Measure of Added Value). A measure of added value, ψ, is a real-valued function on:

$$\bigcup_{(N,v)\in\mathcal{G}(N)} \Big\{(N,v,C) : C \subseteq N\Big\},$$

where $\psi((N,v,C))$ represents the amount of added value in C given the game (N,v). The set of all such measures will be denoted by Ψ.

For notational convenience, we will write $\psi_v^N(C)$ instead of $\psi((N,v,C))$. Furthermore, following common practice in the literature, where the empty set is assumed to has no value (i.e., $v(\emptyset) = 0$), we will assume that the empty set also has no added value, i.e., we will assume that $\psi_v^N(\emptyset) = 0$ for all $(N,v) \in \mathcal{G}(N)$.

As mentioned earlier, our goal is to develop a measure that quantifies the added value in a coalition $C \subseteq N$ based on how much its value *deviates from expectation*. Here, we consider the *expected* value of a coalition to be the result of every member making his or her *average impact*. With this interpretation, every measure of average impact, $\theta \in \Theta$, leads to a different measure of added value, which is:

$$v(C) - \sum_{i\in C} \theta_v^N(i).$$

Now, if we adopt α as our measure of choice for the average impact, we arrive at the following measure, which we call the *Added-Value index*.

Definition 4.2 (Added-Value Index). The *Added-Value index* is the measure of added value, $\chi \in \Psi$, defined for every game $(N,v) \in \mathcal{G}(N)$ and every coalition $C \subseteq N$ as follows:

$$\begin{aligned}\chi_v^N(C) &= v(C) - \sum_{i\in C} \alpha_v^N(i)\\ &= v(C) - \sum_{i\in C} \overline{\phi}_i(N,v).\end{aligned} \tag{24}$$

Note that the Shapley value is not only applicable in *superadditive* games, where $v(C \cup S) \geq v(C) + v(S)$ for every pair of disjoint coalitions $C, S \subseteq N$ (Chalkiadakis et al. 2011), but can also be applied in non-superadditive games.

Figure 2 presents a sample game of four players and illustrates how our index quantifies the added value in each coalition in that game. More specifically, the set of players in this game is $\{1,2,3,4\}$, and the value of every coalition equals the area of the circle that corresponds to that coalition. The figure illustrates how the Shapley value quantifies each player's impact in the entire game (taking all coalition values into consideration) and divides the value of the largest coalition accordingly. The figure also illustrates how our Added-Value index generalizes this idea to quantify the impact of each player not only in the *largest* coalition, but in *each* coalition. Based on this, our index computes the average impact of each player and considers the added value in each coalition to be the difference between the value of that coalition and the sum of the average impacts of the members therein.

Next, we set out to identify a set of properties, or *axioms*, that define our measure, i.e., a set of properties that cannot all be satisfied by any other measure of added value. Such an axiomatic system can be very helpful when determining whether or not to use the measure. In particular, whenever the axioms appear to be desirable, the measure would be a reasonable choice. Conversely, in any setting where the axioms appear to be unattractive, other measures should be used instead. With this in mind, consider the following properties, which are inspired by the Shapley axioms (a discussion of these properties will follow):

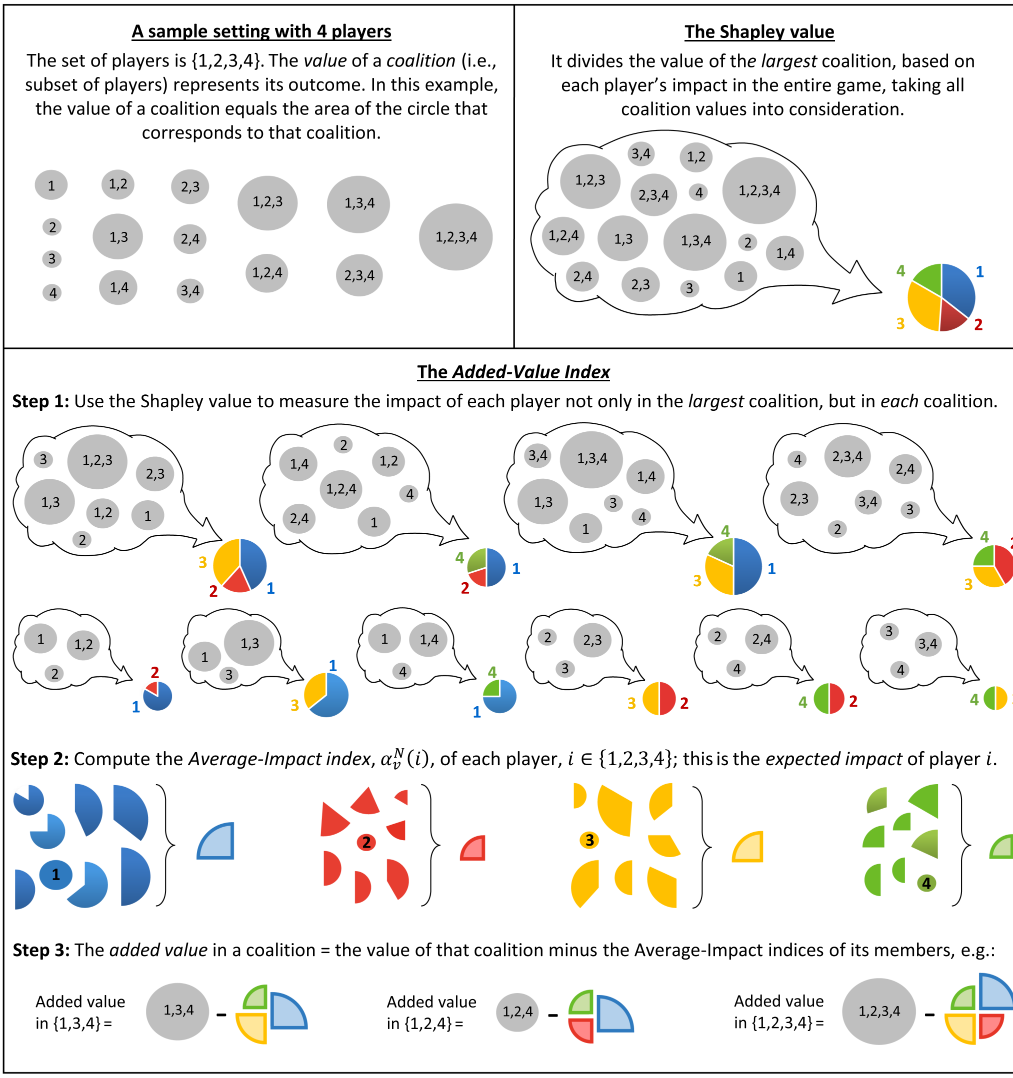

Fig. 2. Given a sample game of four players, the figure illustrates how the *Added-Value index* is calculated and how it relates to the *Shapley value* (Shapley 1953).

P_4. (**Symmetric-Value**): A measure $\psi \in \Psi$ satisfies P_4 if, for every game $(N, v) \in \mathcal{G}(N)$, and every pair of symmetric players i and j in that game, we have: $\psi_v^N(C \cup \{i\}) = \psi_v^N(C \cup \{j\})$ for all $C \subseteq N \setminus \{i, j\}$;

P_5. (**Dummy-Value**): A measure $\psi \in \Psi$ satisfies P_5 if, for every game $(N, v) \in \mathcal{G}(N)$, the following is a game of dummy players: $(N, v - \psi_v^N)$;

P_6. (**Null-Value**): A measure $\psi \in \Psi$ satisfies P_6 if $\psi_v^N(C) = \psi_v^N(C \cup \{i\})$ for every game (N, v) and every null player i in that game;

P_7. (**Normalized-Value**): A measure $\psi \in \Psi$ satisfies P_7 if, for every game $(N, v) \in \mathcal{G}(N)$, we have: $\sum_{C \subseteq N} \psi_v^N(C) = 0$;

P_8. (**Additive-Value**): A measure $\psi \in \Psi$ satisfies P_8 if, for every pair of games, (N, v) and (N, w), and every coalition $C \subseteq N$, we have: $\psi_v^N(C) + \psi_w^N(C) = \psi_{v+w}^N(C)$.

Let us discuss the above properties, starting with P_4. This property states that *replacing a player with another symmetric one does not alter the added value*. This seems natural, because any two symmetric players are essentially identical except for having different names. Thus, it seems strange to claim that they affect the added values differently.

Property P_5 essentially states that *without added value, we are left with a game in which every player is a dummy player*, i.e., we are left with an *additive game*, defined as a game whereby, for all $C, S \subset N$, if $C \cap S = \emptyset$, then $v(C \cup S) = v(C) + v(S)$ (Leyton-Brown and Shoham 2008). This implies that the value of any coalition $C \subseteq N$ in an additive game is simply the sum of the values that the members achieve when each of them $i \in C$ forms its own coalition $\{i\}$. To make the underlying concept clearer, let us put it in the context of our company scenario from the introduction. In an "additive" workplace, whenever a group is formed, each member completely ignores the presence of his or her teammates and acts exactly just as he or she would act when working alone. Arguably, this is how a workplace would appear if all added values were removed from it.

Property P_6 states that *a null player does not affect the added value in a coalition*. In other words, any individual who has no impact on the value of any coalition has no impact on the added value in any coalition. This relates to our interpretation of added value as a *deviation from expectation*; since nothing is produced by, nor expected from, such a player, his or her membership has no impact on the *deviation from expectation*, i.e., has no impact on added value.

Property P_7 also relates to our interpretation of added value as a deviation from expectation; it implies that some groups may yield positive added value (i.e., they exceed our expectations) while others may yield negative added value (i.e., they fall short of our expectations), but on average the groups meet our expectations (i.e., the average of all added values is zero, just as stated by Property P_7), otherwise our expectations would be wrong to start with.

Finally, let us comment on the *Additive-Value* axiom, P_8, which is inspired by Shapley's Additivity axiom. Indeed, even Shapley's choice of that axiom came under criticism until Young (1985) replaced it with a more intuitive one. In fact, this is precisely what we will do later on in Section 5—we will replace our *Additive-Value* axiom with a more intuitive one inspired by Young's work. Still, our *Additive-Value* axiom has the following intuitive implication: scaling a game—by multiplying all coalition values by the same constant—does not change the relative differences between the added values in that game.

We conclude this section with the following theorem, which states that if we view properties P_4 to P_8 as axioms, then those axioms characterize the Added-Value index χ.

Theorem 4.3. *The Added-Value index χ is the only measure in Ψ that satisfies properties P_4 to P_8.*

Proof. We begin by showing that the Added-Value index satisfies the five properties listed in the statement of the theorem.

Claim 2. *The Added-Value index χ satisfies properties P_4 to P_8.*

Proof. Let (N, v) be a game. To prove that χ satisfies P_4, for every pair of symmetric players $i, j \in N$, and every coalition $C \subseteq N \setminus \{i, j\}$, we need to prove that: $\chi_v^N(C \cup \{i\}) = \chi_v^N(C \cup \{j\})$, i.e.,

$$v(C \cup \{i\}) - \sum_{k \in C \cup \{i\}} \overline{\phi}_k(N, v) \quad = \quad v(C \cup \{j\}) - \sum_{k \in C \cup \{j\}} \overline{\phi}_k(N, v). \tag{25}$$

Since i and j are symmetric, then from the definition of symmetry (i.e., Definition 2.2), we know that: $v(C \cup \{i\}) = v(C \cup \{j\}), \forall C \subseteq N \setminus \{i, j\}$. Based on this, as well as the fact that Equation (5) holds (see our proof of Claim 1), we conclude that Equation (25) holds, implying that χ satisfies P_4.

Next, we prove that χ satisfies P_5. Let $\vartheta = v - \chi_v^N$. That is, $\vartheta = \sum_{i \in C} \overline{\phi}_i(N,v)$. We need to prove that every player i in game (N,ϑ) is a dummy player. To this end, it suffices to note that for every $i \in N$ and $C \subseteq N \setminus \{i\}$:

$$\vartheta(C \cup \{i\}) = \sum_{j \in C \cup \{i\}} \overline{\phi}_j(N,v) = \overline{\phi}_i(N,v) + \sum_{j \in C} \overline{\phi}_j(N,v) = \vartheta(\{i\}) + \vartheta(C).$$

Moving on to P_6, we need to prove that $\chi_v^N(C \cup \{i\}) = \chi_v^N(C)$ for every null player i in game (N,v). Since i is a null player in (N,v), then it is also a null player in every sub-game $(C,v) : C \subseteq N$, and so $\overline{\phi}_i(N,v) = 0$. We also know from the definition of a null player that $v(C \cup \{i\}) = v(C)$ for every $C \subseteq N$. As such,

$$\chi_v^N(C \cup \{i\}) = v(C \cup \{i\}) - \sum_{j \in C \cup \{i\}} \overline{\phi}_j(N,v) = v(C) - \sum_{j \in C} \overline{\phi}_j(N,v) = \chi_v^N(C).$$

Let us now consider P_7. We need to show that the following holds:

$$\sum_{C \subseteq N} \chi_v^N(C) = 0. \tag{26}$$

This can be shown as follows:

$$\begin{aligned}
\sum_{C \subseteq N} \chi_v^N(C) &= \sum_{C \subseteq N} \left(v(C) - \sum_{i \in C} \overline{\phi}_i(N,v) \right) \\
&= \sum_{C \subseteq N} v(C) \ - 2^{1-|N|} \sum_{C \subseteq N} \sum_{i \in C} \sum_{S \subseteq N : i \in S} \phi_i(S,v) \\
&= \sum_{C \subseteq N} v(C) \ - \sum_{i \in N} \sum_{S \subseteq N : i \in S} \phi_i(S,v) \\
&= \sum_{C \subseteq N} v(C) \ - \sum_{S \subseteq N} \sum_{i \in S} \phi_i(S,v) \\
&= \sum_{C \subseteq N} v(C) \ - \sum_{S \subseteq N} v(S) \ = \ 0.
\end{aligned}$$

Finally, we deal with P_8. We need to prove that for every pair of games, (N,v) and (N,w), and every $C \subseteq N$, we have: $\chi_v^N(C) + \chi_w^N(C) = \chi_{v+w}^N(C)$. Since ϕ satisfies Shapley's *Additivity* axiom, it follows that $\overline{\phi}_j(N,v+w) = \overline{\phi}_j(N,v) + \overline{\phi}_j(N,w)$. Thus,

$$\begin{aligned}
\chi_{v+w}^N(C \cup \{i\}) &= (v+w)(C \cup \{i\}) \ - \sum_{j \in C \cup \{i\}} \overline{\phi}_j(N,v+w) \\
&= v(C) \ - \sum_{j \in C} \overline{\phi}_j(N,v) \ + \ w(C) \ + \sum_{j \in C} \overline{\phi}_j(N,w) \\
&= \chi_v^N(C) + \chi_w^N(C).
\end{aligned}$$

This concludes the proof of Claim 2. □

Having proved that χ satisfies properties P_4 to P_8, it remains to prove that χ is in fact the only measure in Ψ that satisfies those properties. In other words, assuming that $x \in \Psi$ is a measure satisfying properties P_4 to P_8, we need to show that $x = \chi$. This will be done using the following lemma.

Lemma 4.4. *For every game (N, v), and every coalition $S \subseteq N$, and every constant $\lambda \in \mathbb{R}$, let us denote by $(N, v_{S,\lambda})$ the game where the value of every coalition $C \subseteq N$ is:*

$$v_{S,\lambda}(C) = \begin{cases} \lambda & \text{if } S \subseteq C, \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

Now, if a measure of added value $\psi \in \Psi$ satisfies properties P_4, P_5, P_6, and P_7, then

$$\psi^N_{v_{S,\lambda}}(C) = \begin{cases} \left(1 - 2^{1-|S|}\right) \lambda & \text{if } S \subseteq C, \\ -|C \cap S| \left(2^{1-|S|}\right) \frac{\lambda}{|S|} & \text{otherwise.} \end{cases}$$

Proof. In the game $(N, v_{S,\lambda})$, Property P_5 states that $(N, v_{S,\lambda} - \psi^N_{v_{S,\lambda}})$ is a game of dummy players. In other words, there exist real numbers $(\beta_i)_{i \in N}$ such that:

$$\forall C \subseteq N, \ \ v_{S,\lambda}(C) - \psi^N_{v_{S,\lambda}}(C) = \sum_{i \in C} \beta_i. \tag{28}$$

Now, observe that every $i \in N \setminus S$ is a null player, meaning that: $v_{S,\lambda}(\{i\}) = 0$ and that $\psi^N_{v_{S,\lambda}}(\{i\}) = 0$ (based on Property P_6). Thus, based on Equation (28), we have:

$$\beta_i = 0, \ \ \forall i \in N \setminus S. \tag{29}$$

Furthermore, observe that every pair of players $i, j \in S$ are symmetric, meaning that $v_{S,\lambda}(\{i\}) = v_{S,\lambda}(\{j\})$ and that $\psi^N_{v_{S,\lambda}}(\{i\}) = \psi^N_{v_{S,\lambda}}(\{j\})$ (based on Property P_4). This, as well as Equation (28), imply that there exists a real number, $\beta \in \mathbb{R}$, such that:

$$\beta_i = \beta, \qquad \forall i \in S. \tag{30}$$

Property P_7, as well as Equations (27) to (30), imply that:

$$\left(\sum_{s=1}^{|S|-1} \sum_{C \subseteq N : |C \cap S| = s} (-s\beta) \right) + \sum_{C \subseteq N : S \subseteq C} (\lambda - |S|\beta) = 0.$$

Thus, we have:

$$\beta = \frac{\lambda 2^{1-|S|}}{|S| 2^{|N|-1}}.$$

This, as well as Equations (27) to (30) imply the correctness of Lemma 4.4. □

Recall that we wanted to use Lemma 4.4 to prove that $x = \chi$. To do this, we need to first introduce the notion of a *"carrier game"* (Shapley 1953). In particular, for every coalition $S \subseteq N$, we will denote by (N, v_S) the *carrier game over S*—the game in which the value of a coalition $C \subseteq N$ is:

$$v_S(C) = \begin{cases} 1 & \text{if } S \subseteq C, \\ 0 & \text{otherwise.} \end{cases}$$

Shapley proved that every game (N, v) is a linear combination of carrier games (Shapley 1953). This implies that there exist real numbers $(\lambda_S)_{S \subseteq N, S \neq \emptyset}$ such that, for every $C \subseteq N$, we have:

$$v(C) = \sum_{S \subseteq N, S \neq \emptyset} v_{S,\lambda_S}(C). \tag{31}$$

Now, since x and χ both satisfy properties P_4, P_5, P_6, and P_7, then according to Lemma 4.4:

$$x^N_{v_{S,\lambda_S}} = \chi^N_{v_{S,\lambda_S}}, \quad \forall S \subseteq N, S \neq \emptyset. \tag{32}$$

Based on Equations (31) and (32), as well as the fact that both x and χ satisfy Property P_8, we have:

$$x^N_v = \sum_{S \subseteq N, S \neq \emptyset} x^N_{v_{S,\lambda_S}} = \sum_{S \subseteq N, S \neq \emptyset} \chi^N_{v_{S,\lambda_S}} = \chi^N_v.$$

Since this is true for every game, $(N, v) \in \mathcal{G}(N)$, we conclude that $x = \chi$. This concludes the proof of Theorem 4.3. □

5 AN ALTERNATIVE AXIOMATIZATION OF THE ADDED-VALUE INDEX

The characterization that we introduced in Section 4 for the Added-Value index, $\chi \in \Psi$, uses Property P_8—Additive-Value—which is inspired by Shapley's *Additivity* axiom. Indeed, even Shapley's choice of that axiom came under criticism until Young (1985) replaced it with a more intuitive one based on the notion of *marginal contribution* (see Definition 2.6). Inspired by Young's work, we identify in this section an alternative axiomatization of χ that replaces our *Additive-Value* axiom, P_8, with a more intuitive one based on marginal contribution. To this end, let us introduce the following property:

P_9. (**Marginal-Value**): A measure $\psi \in \Psi$ satisfies P_9 if, for every pair of games (N, v) and (N, w) with the same set of players, and for every coalition $C \subseteq N$, if

$$\forall i \in C, \quad \forall S \subseteq N \setminus \{i\}, \;\; MC_i^S(N, v) = MC_i^S(N, w), \tag{33}$$

then the following holds:

$$\psi_v^N(C) = \psi_w^N(C).$$

We clarify the intuition behind P_9 by continuing on our company scenario from the introduction. Suppose that by the end of Year 1, the company has developed a certain opinion on the added value in a certain group of employees, $C \subseteq N$, where N is the set of all employees in the company. Furthermore, suppose that in Year 2 the performance of every member of C—measured in terms of marginal contribution—remained exactly the same as in Year 1. That is, for every $i \in C$ and every $S \subseteq N \setminus \{i\}$, we have $v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$, where v returns a group's outcome in Year 1 and w returns its outcome in Year 2. Arguably, then, it seems reasonable for the company to keep its opinion regarding the added value from the members of C, regardless of whether the performance (of some or all) of the employees outside C has changed compared to Year 1, i.e., regardless of whether $v(S) = w(S)$. To put it differently, Property P_9 states that the added value in C is a function of the *absolute*, rather than *relative*, impact that the members of C make to various groups of the population (see Table 2 for an illustrative example).

We conclude this section with Theorem 5.1, which essentially states that P_5, P_6, P_7, and P_9 are in fact axioms that characterize the Added-Value index χ. Compared to the previous axiomatization from Section 4, here we replace properties P_8 (Additive-Value) and P_4 (Symmetric-Value) with Property P_9 (Marginal-Value).

Theorem 5.1. *The Added-Value index, $\chi \in \Psi$, is the only measure in Ψ that satisfies properties P_5, P_6, P_7, and P_9 (i.e., Dummy-, Null-, Normalized-, and Marginality-Value).*

Proof. We begin the proof of Theorem 5.1 by showing that the Added-Value index satisfies the four properties listed in the statement of the theorem. As for P_5, P_6, and P_7, we already know from Claim 2 that χ satisfies those properties. Thus, we only need to prove that χ satisfies Property P_9.

Claim 3. *The Added-Value index, $\chi \in \Psi$, satisfies property P_9.*

Proof. Let (N, v) and (N, w) be two games with the same set of players, N, and let $C \subseteq N$ be a coalition for which Equation (33) holds. Based on this, it is easy to show that the following two equations hold[8]:

$$v(C) = w(C). \tag{34}$$

[8]The difference between Equations (33) and (35) is that the former deals with the games (N, v) and (N, w), while the latter deals with all sub-games (T, v) and (T, w) such that $T \subseteq N : i \in T$.

Table 2. Given $N = \{1, 2, 3, 4\}$, Consider Two Sample Games, (N, v) and (N, w), where the Marginal Contribution of Every Player $i \in \{1, 2, 3\}$ to a Coalition $S \subseteq N$ Equals $i \times 10$ in Both Games, while the Marginal Contribution of Player 4 to S Equals 40 in the Game (N, v) and Equals 990 in the Game (N, w)

C	$v(C)$	$w(C)$	$MC_1^C(N, v)$	$MC_1^C(N, w)$	$MC_2^C(N, v)$	$MC_2^C(N, w)$	$MC_3^C(N, v)$	$MC_3^C(N, w)$	$MC_4^C(N, v)$	$MC_4^C(N, w)$
{1}	10	10	10	10						
{2}	20	20			20	20				
{3}	30	30					30	30		
{4}	40	990							40	990
{1, 2}	30	30	10	10	20	20				
{1, 3}	40	40	10	10			30	30		
{1, 4}	50	1,000	10	10					40	990
{2, 3}	50	50			20	20	30	30		
{2, 4}	60	1,010			20	20			40	990
{3, 4}	70	1,020					30	30	40	990
{1, 2, 3}	60	60	10	10	20	20	30	30		
{1, 2, 4}	70	1,020	10	10	20	20			40	990
{1, 3, 4}	80	1,030	10	10			30	30	40	990
{2, 3, 4}	90	1,040			20	20	30	30	40	990
{1, 2, 3, 4}	100	1,050	10	10	20	20	30	30	40	990

Then, according to Property P_9, the added value in any coalition $C \subseteq \{1, 2, 3\}$ is identical in both games, since the marginal contribution of every member of C to every $S \subseteq N$ is identical in both games. This implies that the added value in a coalition is a function of the *absolute*, rather than the *relative*, contribution of each member to all the coalitions in the game. For example, although the marginal contribution of player 1 to {1, 4} equals 10 in both games, this contribution represents 20% of $v(C)$ but only 1% of $w(C)$. As such, a measure satisfying Property P_9 can be thought of as a measure of *absolute*, rather than *relative*, added value.

$$\forall i \in C,\ \forall T \subseteq N : i \in T,\ \forall S \subseteq T \setminus \{i\}, \quad MC_i^S(T, v) = MC_i^S(T, w). \tag{35}$$

Thus, based on the definition of the average Shapley value, $\overline{\phi}_i$ (see Equation (4)), we find that:

$$\forall i \in C,\ \overline{\phi}_i(N, v) = \overline{\phi}_i(N, w). \tag{36}$$

Equations (34) and (36) as well as Definition 4.2—the definition of the Added-Value index—immediately imply the correctness of Claim 3. □

Having proved Claim 3, all that remains is to prove the uniqueness statement in Theorem 5.1. For this, we need to first introduce two new definitions and to prove yet another claim.

Definition 5.2 ($N_v^\dagger$). For every game $(N, v) \in \mathcal{G}(N)$, the set $N_v^\dagger \subseteq N$ is the set of players in (N, v) who are *not* null players. More formally,

$$N_v^\dagger = \{i \in N : \exists C \subseteq N, MC_i^C(N, v) \neq 0\}. \tag{37}$$

Definition 5.3 ((N, v_i)). For every set of players, $N = \{1, 2, \ldots, n\}$, and every $i \in N$, and every characteristic function $v : 2^N \rightarrow \mathbb{R}$, the game (N, v_i) is defined as follows:

$$v_i(C) = v(C \setminus \{i\}), \quad \forall C \subseteq N. \tag{38}$$

Claim 4. *For every game $(N, v) \in \mathcal{G}(N)$, and every $i \in N_v^\dagger$, the following holds:*

$$N_{v_i}^\dagger \subset N_v^\dagger.$$

Proof. Let i be a player in $N^{\dagger}_{v}$. Based on Definition 5.3, we have:

$$\begin{aligned}\forall C \subseteq N : i \in C, \quad MC^C_i(N, v_i) &= v_i(C \cup \{i\}) - v_i(C) \quad \text{(based on Definition 2.6)}\\ &= v_i(C) - v_i(C) \quad \text{(because } i \in C\text{)}\\ &= 0. \end{aligned} \tag{39}$$

$$\begin{aligned}\forall C \subseteq N : i \notin C, \quad MC^C_i(N, v_i) &= v_i(C \cup \{i\}) - v_i(C) \quad \text{(based on Definition 2.6)}\\ &= v(C \cup \{i\} \setminus \{i\}) - v(C \setminus \{i\}) \quad \text{(Definition 5.3)}\\ &= v(C) - v(C) \quad \text{(because } i \notin C\text{)}\\ &= 0. \end{aligned} \tag{40}$$

Equations (39) and (40) imply that i is a null player in the game (N, v_i). Thus, based on Definition 5.2:

$$i \notin N^{\dagger}_{v_i}. \tag{41}$$

However, the following holds for every game (N, v) and any two distinct players, $y, z \in N : y \neq z$:

$$\begin{aligned}\forall C \subseteq N, \; MC^C_z(N, v_y) &= v_y(C \cup \{z\}) - v_y(C)\\ &= v((C \cup \{z\}) \setminus \{y\}) - v(C \setminus \{y\})\\ &= v((C \setminus \{y\}) \cup \{z\}) - v(C \setminus \{y\})\\ &= MC^{C\setminus\{y\}}_z(N, v). \end{aligned} \tag{42}$$

For every $j \in N \setminus N^{\dagger}_{v}$, we know that $j \neq i$ (because $i \in N^{\dagger}_{v}$). Then, based on Equation (42), we have:

$$\forall j \in N \setminus N^{\dagger}_{v}, \quad MC^C_j(N, v_i) = MC^{C\setminus\{i\}}_j(N, v). \tag{43}$$

Furthermore, for every $j \in N \setminus N^{\dagger}_{v}$, we know from Definition 5.2 that j is a null player in (N, v). This, as well as Equation (43), imply that

$$MC^C_j(N, v_i) = 0, \quad \forall j \in N \setminus N^{\dagger}_{v}.$$

This means that j is also a null player in (N, v_i), not just in (N, v). Therefore, based on Definition 5.2, we have:

$$j \notin N^{\dagger}_{v_i}, \quad \forall j \in N \setminus N^{\dagger}_{v}. \tag{44}$$

Equations (41) and (44) imply the correctness of Claim 4. □

Now, we are ready to prove the uniqueness statement in Theorem 5.1. We will do this by showing that, if $x \in \Psi$ is a measure satisfying properties P_5, P_6, P_7, and P_9, then $x = \chi$. The proof will be an inductive one over $|N^{\dagger}_{v}|$; we will prove that:

$$x = \chi, \quad \forall (N, v) \in \mathcal{G}(N) : |N^{\dagger}_{v}| = 0, \tag{45}$$

and that the following holds for every size $s \in \mathbb{N} \setminus \{0\}$:

$$x = \chi, \; \forall (N, v) \in \mathcal{G}(N) : |N^{\dagger}_{v}| < s \quad \Rightarrow \quad x = \chi, \; \forall (N, v) \in \mathcal{G}(N) : |N^{\dagger}_{v}| = s. \tag{46}$$

Step 1: Proving that Equation (45) holds:
Definition 5.2 implies that, for every set of players $N = \{1, 2, \ldots, n\}$, there exists exactly one game $(N, v) \in \mathcal{G}(N)$ such that $|N^{\dagger}_{v}| = 0$; this is the game in which every coalition's value equals zero. In other words, it is the game (N, v^0) where $v^0(C) = 0, \forall C \subseteq N$; this is the only possible game in which every $i \in N$ is a null player. Since both x and χ satisfy Property P_6, and since the empty set

is assumed to always have zero added value, we conclude that:

$$\forall C \subseteq N,\ x^N_{v^0}(C) = \chi^N_{v^0}(C) = 0.$$

Therefore, Equation (45) holds, which is what we wanted to prove in Step 1.

Step 2: Proving that Equation (46) holds:

To this end, assume that $x = \chi, \forall (N, v) \in \mathcal{G}(N) : |N^\dagger_v| < s$ for some $s \in \mathbb{N}$, and let (N, w) be a game in $\mathcal{G}(N)$ such that $|N^\dagger_w| = s$. Since $s > 0$, there exists at least one player in $N^\dagger_w$. Let b be an arbitrary player in $N^\dagger_w$. To prove that Equation (46) holds, it suffices to show that the following equation holds:

$$x^N_w(C) = \chi^N_w(C), \qquad \forall C \subseteq N. \tag{47}$$

To this end, Claim 4 implies that:

$$\left|N^\dagger_{w_b}\right| < \left|N^\dagger_w\right|.$$

Our inductive hypothesis then implies that:

$$x^N_{w_b}(C) = \chi^N_{w_b}(C),\ \ \forall C \subseteq N. \tag{48}$$

Importantly, for every $i \in N \setminus \{b\}$, we know from Equation (42) that:

$$\forall C \subseteq N,\ MC^C_i(N, w_b) = MC^{C\setminus\{b\}}_i(N, w).$$

Therefore,

$$\forall C \subseteq N \setminus \{b\},\ MC^C_i(N, w_b) = MC^C_i(N, w).$$

Based on this, as well as the fact that both x and χ satisfy Property P_9, we find that:

$$\forall C \subseteq N \setminus \{b\},\ x^N_{w_b}(C) = x^N_w(C), \tag{49}$$

$$\forall C \subseteq N \setminus \{b\},\ \chi^N_{w_b}(C) = \chi^N_w(C). \tag{50}$$

Equations (48), (49), and (50) imply that the following holds:

$$x^N_w(C) = \chi^N_w(C), \forall C \subseteq N \setminus \{b\}. \tag{51}$$

Now, since χ satisfies Property P_5, then $(N, w - \chi^N_w)$ is a game of dummy players, meaning that there exist real numbers $(y_i)_{i\in N}$ such that:

$$\forall C \subseteq N,\ \ w(C) - \chi^N_w(C) = \sum_{i\in C} y_i. \tag{52}$$

Likewise, x satisfies Property P_5, and so there exist real numbers $(z_i)_{i\in N}$ such that:

$$\forall C \subseteq N,\ \ w(C) - x^N_w(C) = \sum_{i\in C} z_i. \tag{53}$$

Now, since both x and χ satisfy Property P_7, then:

$$\sum_{C\subseteq N} x^N_v(C) = \sum_{C\subseteq N} \psi^N_v(C).$$

This, as well as Equations (52) and (53), imply that:

$$\sum_{i\in N} y_i = \sum_{i\in N} z_i. \tag{54}$$

We also know that:

$$\begin{aligned} \forall i \in N \setminus \{b\},\ \ y_i &= w(\{i\}) - \chi^N_w(\{i\}) \quad \text{(following Equation (52))} \\ &= w(\{i\}) - x^N_w(\{i\}) \quad \text{(following Equation (51))} \\ &= z_i. \quad \text{(based on Equation (53))} \end{aligned} \tag{55}$$

Equations (54) and (55) imply that:

$$y_i = z_i, \ \forall i \in N. \tag{56}$$

Equations (52), (53), and (56) imply the correctness of Equation (47), which in turn implies the correctness of Equation (46) as discussed earlier. This concludes Step 2 and, consequently, concludes the proof of Theorem 5.1.

6 COMPUTING THE ADDED-VALUE INDEX IN POLYNOMIAL TIME

The Shapley value as well as many other solution concepts from cooperative game theory are computationally challenging when the coalitional game is modeled in the characteristic function form. This is due to the number of possible coalitions to be considered, which grows exponentially with the number of players involved. To handle this computational explosion, a number of alternative representation schemes have been proposed in the literature (see the work by Chalkiadakis et al. (2011) for an overview). *Marginal Contribution networks (or MC-nets)*, due to Ieong and Shoham (2006), belong to the most influential such schemes in the literature (see below). With this scheme, a game is represented by a set of rules, $\mathcal{R}$, each of which is of the form $\mathcal{F} \rightarrow V$, where $\mathcal{F}$ is a propositional formula over the set of players, N, and V is a real number. A coalition C is said to *meet* a given formula $\mathcal{F}$ if and only if $\mathcal{F}$ evaluates to `true` when all Boolean variables corresponding to the players in C are set to `true`, and all Boolean variables corresponding to players outside of C are set to `false`. We write $C \models \mathcal{F}$ to mean that C meets $\mathcal{F}$. In MC-nets, if a coalition C does not meet any rule, then its value is 0. Otherwise, the value of C is the sum of all V from the rules of which formulas $\mathcal{F}$ are met by C. More formally:

$$v(C) = \sum_{\mathcal{F} \rightarrow V \in \mathcal{R} : C \models \mathcal{F}} V.$$

For example, the MC-net where $\mathcal{R} = \{2 \rightarrow 10, 1 \wedge 2 \rightarrow 25\}$ corresponds to the game $(\{1,2\}, v)$, where $v(\{1\}) = 0$, $v(\{2\}) = 10$, and $v(\{1,2\}) = 35$. Intuitively, in this example, the rules mean that whenever player 2 is present in a coalition, the value of that coalition increases by 10, and whenever 1 and 2 are present together in a coalition, its value increases by 25.

Ieong and Shoham (2006) focused on a restricted version of their representation, known as *basic MC-nets*, where $\mathcal{F}$ is made only of conjunctions of positive and/or negative literals:

$$i_1^+ \wedge \ldots \wedge i_{n^+}^+ \wedge \neg i_1^- \wedge \ldots \wedge \neg i_{n^-}^- \rightarrow V,$$

where $N^+ = \{i_1^+, \ldots, i_{n^+}^+\}$ is the set of players from the positive literals, $N^- = \{i_1^-, \ldots, i_{n^-}^-\}$ is the set of players from the negative literals, and $N^+ \cap N^- = \emptyset$. We will often write $\mathcal{F}(N^+, N^-) \rightarrow V$ to denote such a basic rule.

Ieong and Shoham (2006) showed that if the coalitional game is represented by a set of m basic rules $\mathcal{R} = \{\mathcal{F}(N_1^+, N_1^-) \rightarrow V_1, \ldots, \mathcal{F}(N_m^+, N_m^-) \rightarrow V_m\}$, then the Shapley value can be computed in $O(|N|m)$ time. To see why this is the case, recall that one of the well-known properties of the Shapley value is *additivity* (see Section 2). Similarly, the MC-net rules are also additive in the sense that the value of a coalition is computed by adding up the values of all the rules that it meets. Consequently, it can be shown that, when computing the Shapley value of a game represented by a set of rules, $\mathcal{R}$, it is possible to think of each such rule as an independent game on its own—a game represented by just that single rule. Once the Shapley value has been computed for every such game, all that remains is to add up the results, and this would yield the Shapley value of the original game represented by the m basic rules.

Let us consider a game represented by the set of basic MC-net rules $\mathcal{R}$. Following Ieong and Shoham (2006), we can use the fact that the Added-Value index is additive and study each basic rule $(\mathcal{F}(N^+, N^-) \rightarrow V) \in \mathcal{R}$ as an individual game. Next, we can compute the Added-Value index by aggregating the Value-Added indices of all such individual games. In the first step, let us denote by $N^0 = \{i^0_1, \ldots, i^0_{n^0}\}$ all the players in the entire game (i.e., the game represented by the entire set $\mathcal{R}$) that do not belong to $N^+ \cup N^-$. That is, $\{i^0_1, \ldots, i^0_{n^0}\} = N \setminus \{i^+_1, \ldots, i^+_{n^+}, i^-_1, \ldots, i^-_{n^-}\}$. By definition, all these players are *null players* (see Definition 2.3) in the game represented by the single rule $\mathcal{F}(N^+, N^-) \rightarrow V$. Furthermore, we will refer to the players represented by literals in N^+, N^-, and N^0 as *positive*, *negative*, and *neutral* players, respectively. We also note that all positive players are *symmetric* (i.e., they play identical roles in the game; see Definition 2.2). Based on this, the Shapley value of all these players is identical (since the Shapley value is known to reward symmetric players equally; see Section 2). The same argument holds for negative players and for neutral players. Now, for any game, (N, v), represented by a single rule, $\mathcal{F}(N^+, N^-) \rightarrow V$, the following holds (Ieong and Shoham 2006):

— if $N^+ \neq \emptyset$, then

$$\phi_i(N, v) = \begin{cases} V \times \frac{(n^+-1)!n^-!}{(n^++n^-)!} & \text{if } i \in N^+, \\ -V \times \frac{n^+!(n^--1)!}{(n^++n^-)!} & \text{if } i \in N^-, \\ 0 & \text{if } i \in N^0. \end{cases}$$

— if $N^+ = \emptyset$, then

$$\phi_i(N, v) = \begin{cases} -V \times \frac{n^0}{n^-(n^-+n^0)} & \text{if } i \in N^-, \\ V \times \frac{1}{n^-+n^0} & \text{if } i \in N^0. \end{cases}$$

The MC-nets representation is a basic but very popular formalism in the literature (Chalkiadakis et al. 2011). In fact, such an association of numerical weights to propositional logical formulae is a common practice in various domains of knowledge representation (Chevaleyre et al. 2006; Coste-Marquis et al. 2004), and in the languages for combinatorial auctions (Boutilier and Hoos 2001), in particular. In the coalitional games context, MC-nets were extended and generalized in various directions such as more complex, read-once logical formulas (Elkind et al. 2009), to coalitional games with externalities (Michalak et al. 2010; Ichimura et al. 2011), to coalitional games with ordered coalitions (Michalak et al. 2014), to algebraic and multi-terminal Boolean decision diagrams (Aadithya et al. 2011; Sakurai et al. 2011), to hedonic coalitional games (Elkind and Wooldridge 2009), or to games with agent types (Ueda et al. 2011). MC-nets were to fit some particular applications such as recommendation systems (Ieong and Shoham 2006; Kleinberg et al. 2001; Ben-Porat and Tennenholtz 2018) and computing contribution of solvers in an algorithm portfolio (Fréchette et al. 2016; Kotthoff et al. 2018).

As our main result in this section, we will now show that, under the MC-nets representation, the Added-Value index, χ, can be computed in polynomial time.

THEOREM 6.1. *Let (N, v) be a coalitional game represented with a single, basic MC-net rule $\mathcal{F}(N^+, N^-) \rightarrow V$, where $N = \{1, 2, \ldots, n\}$. Then, $\forall C \subseteq N$:*

$$\chi^N_v(C) = \begin{cases} v(C) - \frac{V}{2^{n-n^0-1}} \sum_{k=0}^{n^-} \frac{\binom{n^-}{k}}{\binom{n^++k}{k}} - n^- \sum_{k=1}^{n^-} \frac{\binom{n^--1}{k-1}}{\binom{n^++k}{k} k} & \text{if } N^+ \neq \emptyset, \\ v(C) - \frac{V}{2^{n-1}} \left(n^0 \sum_{k=1}^{n} \frac{\binom{n-1}{k-1}}{k} - n^- \sum_{k=1}^{n^-} \sum_{h=1}^{k-1} \binom{n^--1}{h-1} \binom{n^0}{k-h} \frac{k-h}{kh} \right) & \text{otherwise.} \end{cases}$$

The time-complexity of this formula is $O(|N|^2)$.

Proof. Assume that $N^+ \neq \emptyset$. For every $S \subseteq N : N^+ \nsubseteq S$, the value of every coalition is 0 in the *sub-game* (S, v) (see Definition 2.1). Thus:

$$\overline{\phi}_i(N, v) = 2^{1-n} \sum_{S \subseteq N : i \in S \wedge N^+ \subseteq S} \phi_i(S, v).$$

Consider a positive player, $i^+ \in N^+$. From the result by Ieong and Shoham, we know that for any sub-game (S, v) in which there are exactly k negative players, all positive players, and any number of neutral players, the Shapley value of player i^+ equals:

$$\phi_{i^+}(S, v) = \frac{V(n^+ - 1)!k!}{(n^+ + k)!} \text{ if } N^+ \subseteq S, |N^- \cap S| = k, i^+ \in N^+.$$

There are precisely $2^{n^0}\binom{n^-}{k}$ different subsets of N that consist of k negative players, all positive players, and any number of neutral players. Therefore, for a positive player, $i^+ \in N^+$, we have:

$$\overline{\phi}_{i^+}(N, v) = 2^{1-n} \sum_{k=0}^{n^-} \frac{V(n^+ - 1)!k!2^{n^0}}{(n^+ + k)!} \binom{n^-}{k} = \sum_{k=0}^{n^-} \frac{2^{n^0+1-n}\binom{n^-}{k} V}{\binom{n^+ + k}{k} n^+}.$$

Analogously, consider a negative player, $i^- \in N^-$. We have:

$$\phi_{i^-}(S, v) = \frac{-Vn^+!(k-1)!}{(n^+ + k)!} \text{ if } N^+ \subseteq S,\ |N^- \cap S| = k,\ i^- \in N^-.$$

There are $2^{n^0}\binom{n^- - 1}{k-1}$ different subsets of N that consist of k negative players including i^-, all positive players, and any number of neutral players. Hence, for a negative player $i^- \in N^-$:

$$\overline{\phi}_{i^-}(N, v) = 2^{1-n} \sum_{k=1}^{n^-} \frac{-Vn^+!(k-1)!}{(n^+ + k)!} 2^{n^0} \binom{n^- - 1}{k - 1}.$$

Note that a neutral player has no impact in any sub-game (S, v) since $P \neq \emptyset$. Thus, based on the *Null Player* axiom of the Shapley value (see Section 2), we know that $\overline{\phi}_{i^0}(N, v) = 0$ for every $i^0 \in N^0$. Finally, from Equation (24), we know that:

$$\chi_v^N(C) = v(C) - \left(n^+ \overline{\phi}_{i^+}(N, v) + n^- \overline{\phi}_{i^-}(N, v)\right) \tag{57}$$

$$= v(C) - \frac{V}{2^{n-n^0-1}} \left(\sum_{k=0}^{n^-} \frac{\binom{n^-}{k}}{\binom{n^+ + k}{k}} - n^- \sum_{k=1}^{n^-} \frac{\binom{n^- - 1}{k-1}}{\binom{n^+ + k}{k} k} \right). \tag{58}$$

Having dealt with the case where $N^+ \neq \emptyset$, we now turn our attention to the case where $N^+ = \emptyset$. To this end, consider a neutral player, $i^0 \in N^0$. The Shapley value of player i^0 in a sub-game, (S, v), such that $|S| = k$ and $i^0 \in S$, equals $\frac{V}{k}$. Therefore, we have:

$$\overline{\phi}_{i^0}(N, v) = 2^{1-n} \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{V}{k} \text{ if } i^0 \in N^0.$$

Consider a negative player, $i^- \in N^-$, and a sub-game (S, v) in which there are exactly k players (including i^-), h of which are negative. Here, in a permutation π where i^- is the first negative player, and there is at least one neutral player before i^-, the marginal contribution of i^- to the players that precede him in π is $-V$. Thus, in this sub-game, the Shapley value of player i^- equals $\phi_n(S, v) = \frac{-(k-h)V}{kh}$. Consequently, we have:

$$\overline{\phi}_{i^-}(N, v) = 2^{1-n} \sum_{k=1}^{n^-} \sum_{h=1}^{k-1} \binom{n^- - 1}{h-1} \binom{n^0}{k-h} \frac{-(k-h)V}{kh} \quad \text{if } i^- \in N^-.$$

From Equation (24), we have:

$$\chi_v^N(C) = v(C) - \left(n^0 \overline{\phi}_i^0(N, v) + n^- \overline{\phi}_{i^-}(N, v)\right) \tag{59}$$

$$= v(C) - \frac{V}{2^{n-1}} \left(n^0 \sum_{k=1}^{n} \frac{\binom{n-1}{k-1}}{k} - n^- \sum_{k=1}^{n^-} \sum_{h=1}^{k-1} \binom{n^- - 1}{h-1} \binom{n^0}{k-h} \frac{k-h}{kh}\right). \tag{60}$$

The time-complexity of this formula is $O(|N|^2)$. This concludes the proof of Theorem 6.1. □

7 COMPARISON TO THE LITERATURE

To the best of our knowledge, the literature that is most relevant to our study is that on *group synergy*. For the purpose of this article, the synergy measures in this literature can be divided into two broad categories:

— The literature in the first category studies the synergistic effects in a very specific setting. Typically, the main research question here is how to define and quantify the synergy among *a few* (most often *two*) entities in the setting at hand. As such, the combinatorial issues, which typically emerge in larger groups, are usually not considered.
— The measures in the second category view synergy from a more general, combinatorial perspective, whereby groups of arbitrary sizes are considered.

Our article contributes to the latter category, so that category will be the focus of all but the last subsection. More specifically, Section 7.1 discusses the literature that views synergy from the perspective of Shannon's information theory. Section 7.2 shows how the approach from the previous subsection can be tailored to fit the realm of cooperative game theory. Section 7.3 introduces a measure designed to quantify the interactions among the coalition members. Section 7.4 considers the measure with which the synergy in a coalition C is calculated as the difference between the value of C and the value of the optimal partition of C into strict subsets. Section 7.5 compares our measure of added value against the aforementioned measures of synergy. Finally, Section 7.6 briefly reviews various non-combinatorial concepts of synergy.

7.1 Synergy from the Perspective of Information Theory

In information theory, various sets of random variables can be used to predict a single random variable, called the target variable. In this context, roughly speaking, synergy reflects situations in which the prediction based on a particular set of random variables has relatively low uncertainty compared to the predictions that are based on other sets. When measuring synergy from this perspective, one must take into consideration the fact that the information conveyed by any two variables, 1 and 2, may overlap. To this end, Griffith and Koch (2014) suggested to decompose the information conveyed by $\{1, 2\}$ into:

— the unique information conveyed by 1—we will denote this by $v_u(\{1\})$ where the subscript "u" stands for "unique";
— the unique information conveyed by 2—we will denote this by $v_u(\{2\})$;

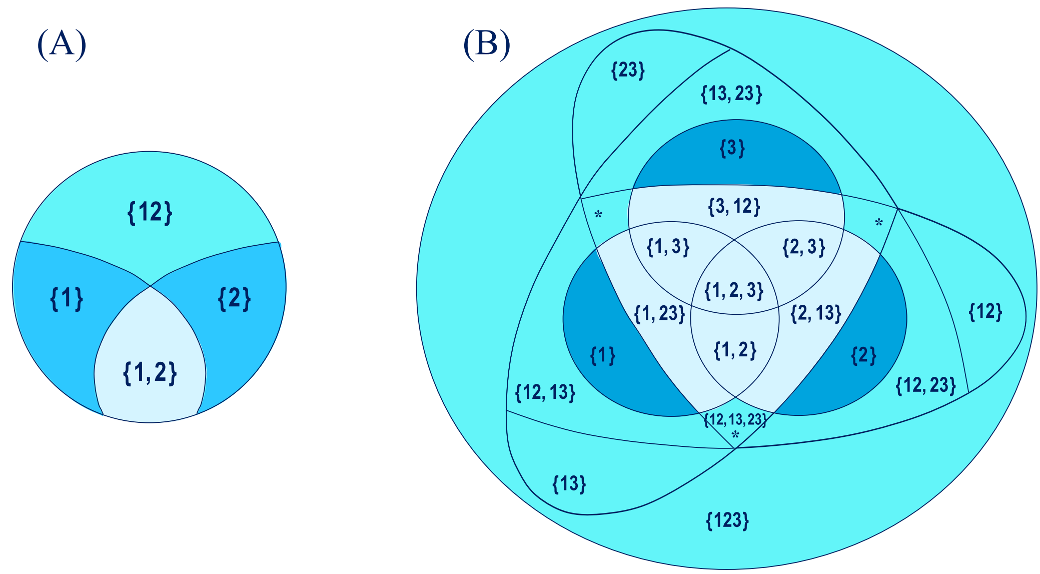

Fig. 3. In information theory, the overlapping information conveyed by any set of variables is explicitly modeled. The left Partial Information diagram illustrates the two-variable case while the right one illustrates the three-variable case. Each region is uniquely identified by its "set notation." In particular, given random variables x and y, the region representing their overlapping information is denoted by $\{x, y\}$, while the synergy between them is denoted by $\{xy\}$. These notations can also be mixed, e.g., $\{x, yz\}$ is the information of x that overlaps with the synergy between y and z. We also note that, to preserve symmetry, region $\{12, 13, 23\}$ in (B) is cut into three separate regions each marked with a star ("⋆"). All those three regions should be considered as a single region. In (A), the synergy between variables 1 and 2 is the region $\{12\}$. In (B), the synergy between 1, 2, and 3 are the regions: $\{12\}$, $\{13\}$, $\{23\}$, $\{123\}$, $\{12, 13\}$, $\{12, 23\}$, $\{13, 23\}$, and $\{12, 13, 23\}$.

- the overlapping information conveyed by both 1 and 2—we will denote this by $v_o(\{1, 2\})$, where the subscript "o" stands for "overlapping"; and
- the synergy from considering both variables jointly—we will denote this by $\psi^{it}(\{1, 2\})$, where the superscript "it" stands for "information theory."

Consequently, we have:

$$v(\{1, 2\}) = v_u(\{1\}) + v_u(\{2\}) + v_o(\{1, 2\}) + \psi^{it}(\{1, 2\}),$$

where $v_u(\{1\}) = v(\{1\}) - v_o(\{1, 2\})$ and $v_u(\{2\}) = v(\{2\}) - v_o(\{1, 2\})$. Thus, the synergy between two variables is:

$$\psi^{it}(\{1, 2\}) = v(\{1, 2\}) - v_u(\{1\}) - v_u(\{2\}) + v_o(\{1, 2\}). \tag{61}$$

This can be generalized to more than just two variables, based on the inclusion-exclusion principle. The above decomposition can be illustrated using *Partial Information diagrams* (Williams and Beer 2010b); the two- and three-variable examples are illustrated in Figure 3(A) and Figure 3(B), respectively.

Commenting on the above interpretation of synergy, Griffith and Koch (2014) argued that the synergy among random variables is basically "*the whole minus the* ***union*** *of its parts*," e.g., the regions $\{12\}$, $\{13\}$, $\{23\}$, $\{123\}$, $\{12, 13\}$, $\{12, 23\}$, $\{13, 23\}$, and $\{12, 13, 23\}$ in Figure 3(B). The authors showed the advantages of this measure of synergy over previous measures from information theory, including "*the whole minus the* ***sum*** *of its parts*" (Chechik et al. 2002), "*the whole minus the* *state-dependent maximum of its parts*" (Williams and Beer 2010a), and the synergy measure based on *correlational importance* (Nirenberg et al. 2001; Latham and Nirenberg 2005).

7.2 Synergy Based on the Harsanyi Dividends

Unlike in information theory, it is typically assumed in coalitional game theory that the values of any two coalitions do not have common elements.[9] This means that, in the two-player case considered in the previous subsection, we now have: $v_o(\{1,2\}) = 0$, $v_u(\{1\}) = v(\{1\})$ and $v_u(\{2\}) = v(\{2\})$, implying that the value of coalition $\{1,2\}$ now consists of the value of $\{1\}$, the value of $\{2\}$, and the synergy between $\{1,2\}$. More formally, the two-players definition of synergy from information theory (see Equation (61)) is adapted as follows (Harsanyi 1958):

$$\psi^{Hd}(\{1,2\}) = v(\{1,2\}) - v(\{1\}) - v(\{2\}). \tag{62}$$

This is known as the *Harsanyi dividends* (Harsanyi 1958) of the coalitional game $(\{1,2\}, v)$, hence the superscript "*Hd*" in ψ^{Hd}. Using the *inclusion-exclusion* principle, Equation (62) can be generalized to measure the synergy in any coalition C of arbitrary size as follows:

$$\psi^{Hd}(C) = \sum_{S \subseteq C} (-1)^{|C|-|S|} v(S). \tag{63}$$

For example, the value of $\{1,2,3\}$ consists of the value of $\{1\}$, the value of $\{2\}$, the value of $\{3\}$, the synergy in $\{1,2\}$, the synergy in $\{1,3\}$, the synergy in $\{2,3\}$, and the synergy in $\{1,2,3\}$. Hence:

$$\psi^{Hd}(\{1,2,3\}) = v(\{1,2,3\}) - v(\{1,2\}) - v(\{1,3\}) - v(\{2,3\}) + v(\{1\}) + v(\{2\}) + v(\{3\}). \tag{64}$$

This resembles the way in which, given three sets of elements, A, B, C, one can compute the number of elements in $A \cup B \cup C$ using the inclusion-exclusion principle as follows; notice the similarity between Equations (64) and (65):

$$|A \cup B \cup C| = |A \cup B \cup C| - |A \cap B| - |A \cap C| - |B \cap C| + |A| + |B| + |C|. \tag{65}$$

Importantly, the key difference between ψ^{Hd} and our measure of added value, χ, is that the former is built upon the assumption that the cooperation among a group of players encompasses the cooperation among every subset of this group. As such, ψ^{Hd} does not take into consideration how the members of coalition C perform outside of C. In contrast, our measure χ quantifies the difference in value over the average impacts of all members, where those averages reflect each member's performance *not only inside* C but also *outside* of C.

7.3 Synergy Based on the Interaction Index

The *interaction index* quantifies the added value that a group of players brings to the game when those players cooperate in various coalitions. It was first proposed by Owen (1972), whose idea was then rediscovered by Murofushi and Soneda (1993) in the field of fuzzy control systems. Formally, the interaction index between two players $i, j \in N$, is defined as follows, where "*ii*" stands for the "interaction index":

$$\psi^{ii}(\{i,j\}) = \sum_{T \subset N \setminus \{i,j\}} \frac{(|N| - |T| - 2)!|T|!}{(|N| - 1)!} \Big[v(T \cup \{i,j\} - v(T \cup \{i\}) - v(T \cup \{j\}) + v(T) \Big].$$

Intuitively, this interaction index is a weighted average of the synergy (as defined in Equation (63)) from the cooperation between i and j in all coalitions that contain both of these players. Inspired

[9] We note that there exist games in which this is not necessarily the case. For instance, when payoffs are expressed in terms of goals, then if one coalition achieves a certain goal, other coalitions may no longer be able to achieve that goal (Sandholm et al. 1999).

by the Shapley value, Grabisch and Roubens (1999) generalized the notion of the interaction index to coalitions of arbitrary sizes as follows:

$$\psi^{ii}(C) = \sum_{T \subseteq N \setminus C} \left(\frac{(|N| - |T| - |C|)!|T|!}{(|N| - |C| + 1)!} \sum_{S \subseteq C} (-1)^{|C|-|S|} v(S \cup T) \right). \tag{66}$$

In addition to game theory and fuzzy systems, this generalization was also extensively studied in various other fields, including: multi-criteria decision-making, aggregation function theory, and statistics and data analysis. A comprehensive introduction to this vivid body of literature can be found in the work by Marichal and Mathonet (2008).

To the best of our knowledge, the interaction index is the only measure of synergy characterized by the axiomatic approach. In particular, Grabisch and Roubens (1999) proposed four axioms:

(1) The *Linearity* axiom implies that the measure is a weighted average of the coalition values; thus, it can be considered a much stronger version of P_8 (Additive-Value).
(2) The *Dummy* axiom implies that the synergy in a coalition containing a dummy player equals zero. This axiom stands in contradiction with P_6 (Null-Value).
(3) The *Symmetry* axiom, similarly to our P_4 (Symmetric-Value), implies that the measure does not depend on players' names.
(4) The *Recursive* axiom directly specifies the synergy in a group based on the synergies in the subgames without a particular player.

Finally, the authors assume that the restriction to singletons corresponds to the Shapley value of the game. Among these axioms, our Added-Value index satisfies only Linearity and Symmetry.

7.4 Synergy as the Surplus Over the Optimal Partition into Strict Subsets

The concept of synergy plays a prominent role in business, especially in the management of large companies. In this context, synergy is understood as an improvement caused by the implementation of more efficient methods of collaboration. For instance, the divisional structure of most major companies systematically evolves in the hope of improving efficiency, or, in other words, achieving synergy. From this point of view, the existence of an organizational unit makes sense only if it is impossible to partition the unit in a better way (Beneke et al. 2007). A widely known critical analysis of this approach to synergy in the business context can be found in Goold and Campbell (1998).

The above approach to synergy has the following interpretation in the context of coalitional games. The synergy in a coalition, C, equals the value of C minus the value of the optimal partition of C into strict subsets. More formally, if we denote by $\mathcal{P}^C$ the set of partitions of C, then for all $C \subseteq N$, we have:

$$\psi^{sop}(C) = v(C) - \max_{P \in \mathcal{P}^C : P \neq \{C\}} \left(\sum_{S \in P} v(S) \right), \tag{67}$$

where "*sop*" stands for "Surplus over Optimal Partition." Based on this equation, a coalition C achieves positive (negative) synergy if the value of C is greater (smaller) than the value of its best partition of C into strict coalitions.

A computational study of certain aspects of coalitional games based on this approach to synergy was proposed in the computer science literature by Conitzer and Sandholm (2006) and Ohta et al. 2009. In this context, the key issue is to compute the optimal partition of a set of agents that is called the coalition structure generation problem. This problem is NP-Complete (Sandholm et al. 1999) and various algorithms have been proposed in the literature to tackle this complexity (Rahwan et al. 2015).

7.5 Comparison of the Combinatorial Measures of Synergy

In this subsection, we discuss some of the differences between our Added-Value index (i.e., χ from Section 4) and other combinatorial measures of synergy presented earlier in this section, namely:

- the *information-theory* measure, i.e., ψ^{it} from Section 7.1;
- the *Harsanyi-dividends* measure, i.e., ψ^{Hd} from Section 7.2;
- the *Interaction-Index* measure, i.e., ψ^{ii} from Section 7.3;
- the measure of synergy as the *surplus over the optimal partition*, i.e., ψ^{sop} from Section 7.4.

We will first comment on some fundamental mathematical properties of the above measures and then consider two sample games—one with synthetic data and another based on real-life citation data from Google Schola—to compare those measures with the Added-Value index.

7.5.1 Some Fundamental Mathematical Properties. Starting with the information-theory measure, in our context where coalitions do not overlap, the synergy in any given coalition equals the value of that coalition minus the sum of the values attained when the members form *singletons* (i.e., single-member coalitions). More formally:

$$\psi^{it}(C) = v(C) - \sum_{i \in C} v(\{i\}).$$

As can be seen, $\psi^{it}(C)$ depends solely on the value of C and the values of the singletons formed by the members of C. The table below presents the information-theory synergies in an *additive*[10] five-player game called *Game 1* and demonstrates how even a slight alternation in the values of the singletons (in green, $\varepsilon > 0$) can change all the positive synergies (in blue) into negative ones (in red), and *vice versa.*

	Game 1			*Game 2*			*Game 3*		
C	$v(C)$	$\psi^{it}(C)$	$\chi_v^N(C)$	$v(C)$	$\psi^{it}(C)$	$\chi_v^N(C)$	$v(C)$	$\psi^{it}(C)$	$\chi_v^N(C)$
$\lvert C\rvert = 1$	1	0	0	$1 - \varepsilon$	0	$-15/16 \times \varepsilon$	$1 + \varepsilon$	0	$15/16 \times \varepsilon$
$\lvert C\rvert = 2$	2	0	0	2	$2 \times \varepsilon$	$2/16 \times \varepsilon$	2	$-2 \times \varepsilon$	$-2/16 \times \varepsilon$
$\lvert C\rvert = 3$	3	0	0	3	$3 \times \varepsilon$	$3/16 \times \varepsilon$	3	$-3 \times \varepsilon$	$-3/16 \times \varepsilon$
$\lvert C\rvert = 4$	4	0	0	4	$4 \times \varepsilon$	$4/16 \times \varepsilon$	4	$-4 \times \varepsilon$	$-4/16 \times \varepsilon$
$C = N$	5	0	0	5	$5 \times \varepsilon$	$5/16 \times \varepsilon$	5	$-5 \times \varepsilon$	$-5/16 \times \varepsilon$

The same effect can also be observed for the Added-Value index, $\chi_v^N(C)$, however, the values of synergies are substantially different. In particular, in the baseline (i.e., Game 1), the value of the Average-Impact index is 1 for all the players, since the Shapley value of each player is 1 in every subgame. As a result, the Added-Value index is 0 for every coalition. In Game 2, where the value of the singletons is changed to $1 - \varepsilon$, the value of the Average-Impact index becomes $1 - \frac{\varepsilon}{16} < 1$. Intuitively, the $-\varepsilon$ change in the value of coalition $C = \{i\}$ has a negative impact (of $-\varepsilon$) on the Shapley value in only one subgame, namely $(\{i\}, v)$, and the Shapley value of player i in each of the other 15 subgames remains the same. Therefore, the Average-Impact index changes by $-\frac{\varepsilon}{16}$. Since, for every coalition $|C| \geq 2$, the sum of the Average-Impact indices is smaller than the value of this coalition, the synergy according to the Added-Value index is always positive but smaller than the synergy according to the information-theory measure. This is because, when computing $\psi^{it}(C)$ for coalition $|C| \geq 2$ in Game 2, we subtract $|C| \times (1 - \varepsilon)$, but when computing $\chi_v^N(C)$, we

[10]Recall that a coalitional game is said to be *additive* if, for all $C, S \subset N$, we have $v(C \cup S) = v(C) + v(S)$ if $C \cap S = \emptyset$ (Leyton-Brown and Shoham 2008).

subtract $|C| \times \left(1 - \frac{\varepsilon}{16}\right)$. For this reason, $\chi_v^N(C)$ deems the value of singletons, $1 - \varepsilon$, to be below the expectation; hence, the negative synergy. As for the Harsanyi-dividends measure ψ^{Hd}, we note that its definition from Equation (63) can be rewritten as a recursive formula as follows, bearing in mind that $\psi^{Hd}(\emptyset) = 0$:

$$\psi^{Hd}(C) = v(C) - \sum_{S \subset C} \psi^{Hd}(S). \tag{68}$$

In words, the synergy in a coalition C according to ψ^{Hd} is simply the value of C minus the sum of all the synergies in the strict subsets of C. While this definition of synergy is elegant mathematically, it may produce results that are somewhat non-intuitive and very different from the results obtained by the information-theory measure. To illustrate this point, let us compare the two measures given *Game 1* and a modified version of it called *Game 4* (the difference between Game 1 and Game 4 is highlighted in green; positive synergies are highlighted in blue; negative synergies are highlighted in red).

	Game 1				*Game 4*			
C	$v(C)$	$\psi^{it}(C)$	$\psi^{Hd}(C)$	$\chi_v^N(C)$	$v(C)$	$\psi^{it}(C)$	$\psi^{Hd}(C)$	$\chi_v^N(C)$
$\lvert C\rvert = 1$	1	0	1	0	1	0	1	3/32
$\lvert C\rvert = 2$	2	0	0	0	1.5	−0.5	−0.5	−10/32
$\lvert C\rvert = 3$	3	0	0	0	3	0	1.5	9/32
$\lvert C\rvert = 4$	4	0	0	0	4	0	−3	12/32
$C = N$	5	0	0	0	5	0	5	15/32

As can be seen, in an additive game such as *Game 1*, both measures are identical except for the way they view singletons. However, the situation completely changes as soon as we introduce some alteration to this additive game, as is the case with *Game 4*. In this latter game, while both measures agree that the synergy in a coalition of size 2 is negative (based on the fact that the members can do better when working alone), the measures diverge in their evaluation of synergy in larger coalitions. To see why this is the case, let us first analyze the coalitions of size 3. Since the value of any such coalition, $\{i, j, k\}$, equals the sum of the values achieved when the members work separately, the information-theory measure, ψ^{it}, returns 0. In contrast, the synergy according to the Harsanyi-dividends measure, ψ^{Hd}, is 1.5 as detailed below:

$$\begin{aligned}\psi^{Hd}(\{i,j,k\}) &= v(\{i,j,k\}) - \psi^{Hd}(\{i,j\}) - \psi^{Hd}(\{i,k\}) - \psi^{Hd}(\{j,k\}) - \psi^{Hd}(\{i\}) - \psi^{Hd}(\{j\}) - \psi^{Hd}(\{k\}) \\ &= 3 + 0.5 + 0.5 + 0.5 - 1 - 1 - 1 \\ &= 1.5.\end{aligned}$$

In words, while the players do not generate any surplus over the sum of the corresponding singleton values (as captured by ψ^{it}), the synergy is deemed positive according to ψ^{Hd} primarily because of the negative synergies in the coalitions of size 2 (i.e., the three players seem to overcome such negative synergies, which explains why they end up being assigned a positive synergy). As for any coalition of size 4, again the members together do not generate any surplus over the sum of their singleton values; hence, ψ^{it} assigns no synergy to them. However, the synergy is deemed negative according to ψ^{Hd} for the reasons mentioned earlier. Those same reasons also explain why the synergy becomes once again positive in the coalition of size 5 according to ψ^{Hd} and remains 0 according to ψ^{it}. This is why the sign of ψ^{it} keeps alternating as we increase the coalition size; see the table above.

Having discussed ψ^{it} and ψ^{Hd}, let us now discuss the interaction-index measure, ψ^{ii}. This measure is actually very closely related to ψ^{Hd} as evident from their corresponding definitions in Equation (63) and Equation (66), respectively. By looking at these equations, one can see that they both contain $\sum_{S\subseteq C}(-1)^{|C|-|S|}$, which alternates between positive and negative. In fact, ψ^{Hd} is used as the cornerstone for ψ^{ii}. Roughly speaking, $\psi^{ii}(C)$ can be thought of as a weighted average of the synergies among the members of C (measured according to ψ^{Hd}) taken over all the coalitions that contain *all* the members of C; see Equation (66) for a more formal definition.

Let us also comment on the Added-Value index in Game 4. The -0.5 change in the value of the coalitions of size 2 means that the Shapley value in all subgames $(C, v) : |C| = 2$ changes from 1 to $1.5/2 = 0.75$ for both players. Since, out of all 16 coalitions that a player i belongs to, there are exactly 6 coalitions of size two, the Average-Impact index changes by $-6 \times 0.25/16 = -3/32$ when compared to Game 1, which yields the results in the table above.

Finally, let us discuss ψ^{sop}—the measure whereby synergy is interpreted as the surplus over the optimal partition of the coalition into strict subsets. Let us examine the synergy according to this measure given *Game 1*, and given a modified version of it called *Game 5* (the difference between Game 1 and Game 5 is highlighted in green; positive synergies are highlighted in blue; negative synergies are highlighted in red):

	Game 1				*Game 5*			
C	$v(C)$	$\psi^{it}(C)$	$\psi^{sop}(C)$	$\chi_v^N(C)$	$v(C)$	$\psi^{it}(C)$	$\psi^{sop}(C)$	$\chi_v^N(C)$
$\|C\| = 1$	1	0	1	0	1	0	1	$-3/32$
$\|C\| = 2$	2	0	0	0	2.5	0.5	0.5	$10/32$
$\|C\| = 3$	3	0	0	0	3	0	-0.5	$-9/32$
$\|C\| = 4$	4	0	0	0	4	0	-1	$-12/32$
$C = N$	5	0	0	0	5	0	-1	$-15/32$

In *Game 1*, which is an additive game, the measure is identical to ψ^{it} except for the way it perceives singletons. However, this is not the case in *Game 5*, where the coalitions of size 2 are the only ones whose values are greater than the sum of their members' singleton values. As such, the synergy according to ψ^{sop} is deemed positive in every coalition of size 2 and negative in every larger coalition (because the division of any such coalition into a partition that contains pairs will be better than the coalition itself).

As for the Added-Value index in Game 5, its values are exactly the same as in Game 4 but with the opposite sign.

Finally, in Table 3, we compare all five measures in terms of our axiomatic properties proposed in Section 4. As can be seen:

— P_4 (Symmetric-Value) is satisfied by all measures of synergy.
— P_5 (Dummy-Value) is violated by ψ^{Hd}, ψ^{ii}, and ψ^{sop}, as can be seen in Game 1: for all three measures, we have $\psi(C) = 1$ if $|C| = 1$ and $\psi(C) = 0$ if $|C| = 2$, which implies $(v - \psi)(C) = 0$ if $|C| = 1$ and $(v - \psi)(C) = 2$ if $|C| = 2$; thus, $(v - \psi)$ is not a dummy-game.
— P_6 (Null-Value) is violated by ψ^{Hd} and ψ^{ii}. In fact, it can be proven that, according to these measures, adding a null-player to a coalition changes its synergy to zero.
— P_7 (Normalized-Value) is violated by all measures other than Added-Value index. To see that, consider again Game 1—the sum of synergies equals 5 for ψ^{Hd}, ψ^{ii}, and ψ^{sop}. However, Game 4 shows that the sum of synergies according to ψ^{it} also equals 5.
— P_8 (Additive-Value) is satisfied by all measures of synergies except ψ^{sop}, which is the only non-linear measure. To see that, consider two games, $(\{i, j, k\}, v)$ and $(\{i, j, k\}, w)$, such

Table 3. Comparisons of the Added-Value Index Against the Other Synergy Measures in Term of Axioms $P_4, \ldots, P_8$

	Information -theory	*Harsanyi -dividends*	*Interaction -index*	*Surplus over opt. partition*	***Added-Value index***
	ψ^{it} Section 7.1	ψ^{Hd} Section 7.2	ψ^{ii} Section 7.3	ψ^{sop} Section 7.4	χ_v^N Section 4
P_4 (Symmetric-Value)	✓	✓	✓	✓	✓
P_5 (Dummy-Value)	✓	×	×	×	✓
P_6 (Null-Value)	✓	×	×	✓	✓
P_7 (Normalized-Value)	×	×	×	×	✓
P_8 (Additive-Value)	✓	✓	✓	×	✓

In particular, for each measure of synergy from the literature, the table indicates whether a given axiom is (✓) or is not (×) satisfied.

that $v(\{i,j\}) = v(\{k\}) = w(\{i\}) = w(\{j,k\}) = 1$, and $v(C) = 0$, $w(C) = 0$ for other coalitions. Clearly, we have $\psi^{sop}(\{i,j,k\}) = -2$ for v, w and $v + w$, which violates P_8.

As we proved in Theorem 4.3, the Added-Value index is the only measure that satisfies all of the above five axioms.

7.5.2 Two Further Examples. Having discussed some basic mathematical properties of the main synergy measures from the literature, we are now ready to demonstrate how they differ from our Added-Value index. Let us first revisit the 4-player example that was presented in Figure 1. For the reader's convenience, the same illustration is repeated in Figure 4, except that the histogram of each player is now annotated with the *Average-Impact index* of that player. Recall that this figure specifies the revenue generated by each player in every coalition that he/she may belong to. Looking at this figure, one can see that the game intuitively represents a scenario in which:

— Player 1 prefers to work in a team as opposed to working alone. Furthermore, of all the other individuals that player 1 can work with, he/she strongly prefers to work with player 4.
— Player 2's preferences and revenues are identical to those of player 1. In fact, one can verify that the two players are *symmetric*; see Definition 2.2.
— Player 3 dislikes working in teams, and his/her revenue decreases with the team size.
— Player 4 is social, and hence his/her revenue increases with the team size.

In this game, the value of every group is the sum of the revenues of its members. For instance, we have: $v(\{2,4\}) = 100 + 40 = 140$ and $v(\{1,2,3\}) = 40 + 40 + 30 = 110$. Table 4 reports the values of all the coalitions and compares our Added-Value index with the other measures of synergy.

Before commenting on those results, let us first comment on the *Average-Impact index* of each player, i.e., $\alpha_v^N(i)$ from Section 3, which is used as the cornerstone for our Added-Value index. Recall that the former index reflects the average performance of each player in various coalitions. For the game illustrated in Figure 4, the Average-Impact indices are as follows: $\alpha_v^N(1) = 47.5$; $\alpha_v^N(2) = 47.5$; $\alpha_v^N(3) = 51.67$; $\alpha_v^N(4) = 63.33$. This means that the inclusion of, say, player 1 increases a coalition's value by an average of 47.5. Note that this is different from the average revenue of player 1 in Figure 4. More specifically, looking at this figure, one can see that the average revenue of player 1 is in fact 60, not 47.5. This is because the Average-Impact index partially attributes to player 1 any increase (or decrease) in the other players' revenues when they join player 1. For example, looking at the coalition values in Table 4, one can see that $v(\{2,3,4\}) = 190$ and $v(\{1,2,3,4\}) = 280$, meaning that when player 1 joins $\{2,3,4\}$ the value increases by 90. This

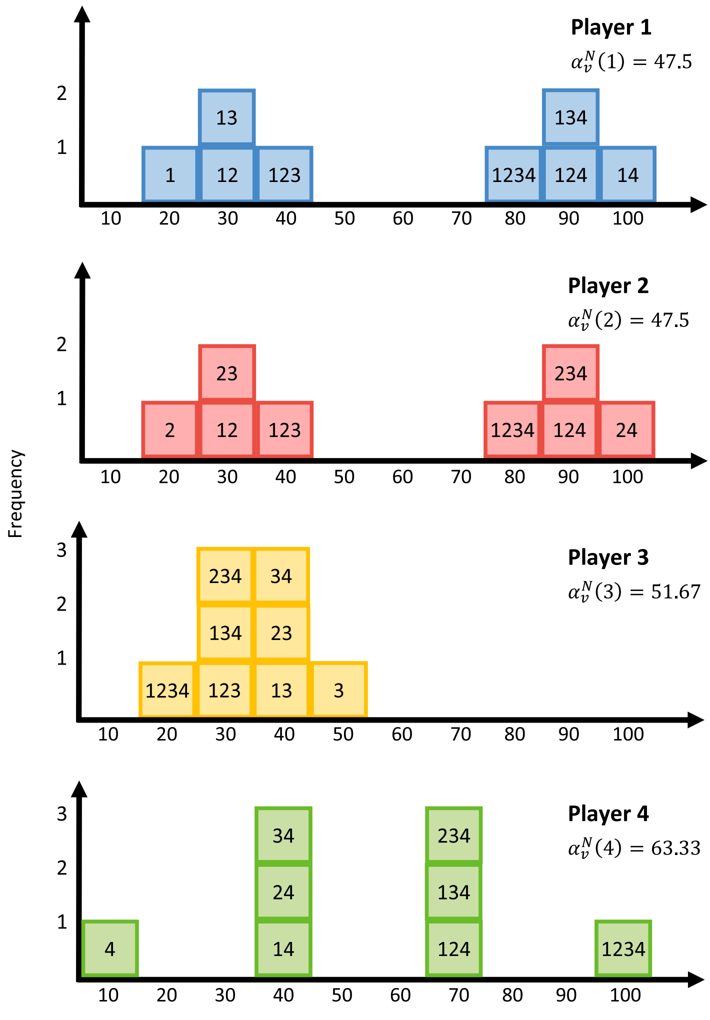

Fig. 4. A four-player game where, in each coalition, every member generates a revenue, and the value of the coalition is then equal to the sum of the revenues of its members. This game is represented by four histograms, each specifying the revenue of a particular player in various coalitions. For example, the revenue of player 1 equals 20 when working alone, 30 when working in either $\{1, 2\}$ or $\{1, 3\}$, 40 when working in $\{1, 2, 3\}$, 80 when working in $\{1, 2, 3, 4\}$, 90 when working in either $\{1, 2, 4\}$ or $\{1, 3, 4\}$, and 100 when working in $\{1, 4\}$ (this is indeed a histogram, since the height of each bar represents the frequency of any given revenue, e.g., there is 1 coalition in which player 1's revenue is 20, and 2 coalitions in which player 1's revenue is 30, etc.). Based on this, the value of any given coalition can be computed by summing the revenues of the members, e.g., $v(\{2, 3\}) = 30 + 40 = 70$ and $v(\{1, 2, 4\}) = 90 + 90 + 70 = 250$. Note that the histogram of each player is annotated with the *Average-Impact index* of that player; see Section 3 for more details on the Average-Impact index, α_v^N.

Table 4. Given the Game Defined by the Revenues in Figure 4, the Table Compares Our Added-Value Index, χ_v^N, Against the Other Synergy Measures, Which Are: ψ^{it} (the Measure from Information Theory); ψ^{Hd} (the Measure Based on the Harsanyi Dividends); ψ^{ii} (the Measure Based on the Interaction Index), and ψ^{sop} (the Measure Whereby Synergy Is the Surplus Over the Optimal Partition)

S	$v(S)$	$\psi^{it}(S)$ Section 7.1	$\psi^{Hd}(S)$ Section 7.2	$\psi^{ii}(S)$ Section 7.3	$\psi^{sop}(S)$ Section 7.4	$\chi_v^N(S)$ Section 4
$\{1\}$	20	0	20	65	20	−27.5
$\{2\}$	20	0	20	65	20	−27.5
$\{3\}$	50	0	50	46.67	50	−1.67
$\{4\}$	10	0	10	103.33	10	−53.33
$\{1,2\}$	60	20	20	0	20	−35
$\{1,3\}$	70	0	0	−10	0	−29.17
$\{1,4\}$	140	110	110	80	110	29.17
$\{2,3\}$	70	0	0	−10	0	−29.17
$\{2,4\}$	140	110	110	80	110	29.17
$\{3,4\}$	80	20	20	0	20	−35
$\{1,2,3\}$	110	20	0	0	0	−36.67
$\{1,2,4\}$	250	200	−40	−40	90	91.67
$\{1,3,4\}$	190	110	−20	−20	0	27.5
$\{2,3,4\}$	190	110	−20	−20	0	27.5
$\{1,2,3,4\}$	280	180	0	0	−20	70

Synergies are highlighted in blue, while antergies are highlighted in red.

increase is not just the result of player 1's revenue in $\{1, 2, 3, 4\}$ (which is 80; see Figure 4), but is also the result of an overall boost in the other players' revenue when player 1 joins them. Intuitively, it is such changes in the other players' revenues that cause the discrepancy between player 1's Average-Impact index and player 1's average revenue.

Having calculated the Average-Impact indices, $\alpha_v^N(i)$, we can now easily compute the Added-Value index, $\chi_v^N(C)$, by subtracting from $v(C)$ the sum of the Average-Impact indices of the members. More formally, we have: $\chi_v^N(C) = v(C) - \sum_{i \in C} \alpha_v^N(i)$. Let us discuss a number of cases that illuminate the intuition behind this index:

- $\chi_v^N(\{1, 2, 3\}) = -36.67$. Looking at Figure 4, one can immediately see why the added value in this coalition is deemed negative. Specifically, the histograms of players 1 and 2 clearly show that their performance drops below average in the absence of player 4. Likewise, the histogram of player 3 shows that their performance drops below average when working in $\{1, 2, 3\}$. Crucially, this conclusion can only be reached taking into consideration the performance of the members of $\{1, 2, 3\}$ not only *inside* the coalition, but also *outside* of it (see how none of the other measures returns a negative synergy for this coalition).
- $\chi_v^N(\{1, 2, 4\}) = 91.67$. One can immediately see from Figure 4 why the formation of this coalition yields positive added value. In particular, the histograms of the members clearly show that their performance in this coalition is much greater than average. Despite this fact, the measures ψ^{Hd} and ψ^{ii} return a negative value.
- $\chi_v^N(\{1, 2, 3, 4\}) = 70$. Looking at Figure 4, one can see that in this coalition the performance of players 1, 2, and 4 is greater than average, although this is not the case with player 3. The figure also shows that the improvements in the performance of players 1, 2, and 4 outweigh the drop in player 3's performance. As such, the added value is deemed positive,

but its magnitude is smaller than that in $\{1, 2, 4\}$ (where all the members perform better than average). This is despite the fact that $v(\{1, 2, 4\})$ is actually smaller than $v(\{1, 2, 3, 4\})$. More generally, the value of a coalition is not necessarily reflective of the added value therein. Again, of all the other measures, the only one that returns a positive synergy is ψ^{it}.

- $\chi_v^N(\{1\}) = -27.5$. To see why the added value in this singleton is negative, recall that in this game the Average-Impact index of player 1 equals 47.5. This means that, on average, player 1 is capable of increasing the value of a coalition by 47.5. Viewed from this perspective, the performance of player 1 when working alone seems below average, hence the negative added value in $\{1\}$. In contrast, the other measures return either 0 or return $v(\{1\})$.

Having discussed a synthetic example, let us now discuss a real-life example in which players represent scientists and coalitions represent collaborations. To this end, we searched Google Scholar for a group of four scientists who "*formed various coalitions*," i.e., collaborated in various combinations. We identified a group of four scientists who formed 13 out of the possible 15 coalitions. These scientists are:

- Diego Calvanese—Faculty of Computer Science of the Free University of Bozen-Bolzano;
- Maurizio Lenzerini—School of Engineering in Computer Science, Sapienza University of Rome;
- Giuseppe De Giacomo—Department of Computer, Control, and Management Engineering Antonio Ruberti at Sapienza University of Rome;
- Riccardo Rosati—Department of Computer, Control, and Management Engineering Antonio Ruberti at Sapienza University of Rome.

A full list of their corresponding publications is listed in Table 6 in the appendix. The way we measured the impact of any given paper was inspired by the work of Sinatra et al. 2016, who quantify impact based on the number of citations that the paper accumulates 10 years after its publication. We calculate a similar measure but focus on 5 rather than 10 years; we refer to this as the "5-year" impact of the paper. Based on this, for each coalition (i.e., subset of the above four scientists), we define the value of that coalition as the *average* 5-year impact, taken over all the papers authored by that coalition and published before the year 2014 (we do not consider any papers published in 2014 or afterwards, to allow for citations to accumulate 5 years after publication). The resulting coalition values are presented in Figure 5. For the two coalitions that have no values, i.e., $\{C, L, R\}$ and $\{L, R\}$, their values were assumed to be equal to the average of all coalitions with known values that are connected to them in the subset lattice (see Figure 5).

The synergies according to the different measures are presented in Table 5. Here, the values of singletons are high enough that the synergies in all remaining coalitions are deemed negative according to the information-theory measure (this resembles the situation in *Game 3* from the previous subsection). Furthermore, the singletons constitute the best partition of every other coalition, meaning that the measure of synergy as a surplus over the optimal partition is always negative (these results resemble those obtained in *Game 5* from the previous subsection). As for the Harsanyi-dividends measure, we observe the pattern of alternating signs of synergy for each coalition size (this is consistent with our analysis of *Game 4* from the previous subsection). Moreover, this last observation also holds for most coalitions as far as the interaction-index measure is concerned; this is due to the similarity between this measure and the Harsanyi-dividends measure (as discussed in the previous subsection).

Finally, let us comment on the results of the Added-Value index in Table 5.

- *Singletons*: our measure also recognizes that the singletons perform well in this game (this resembles the situation in *Game 3* from the previous subsection). In particular, the values

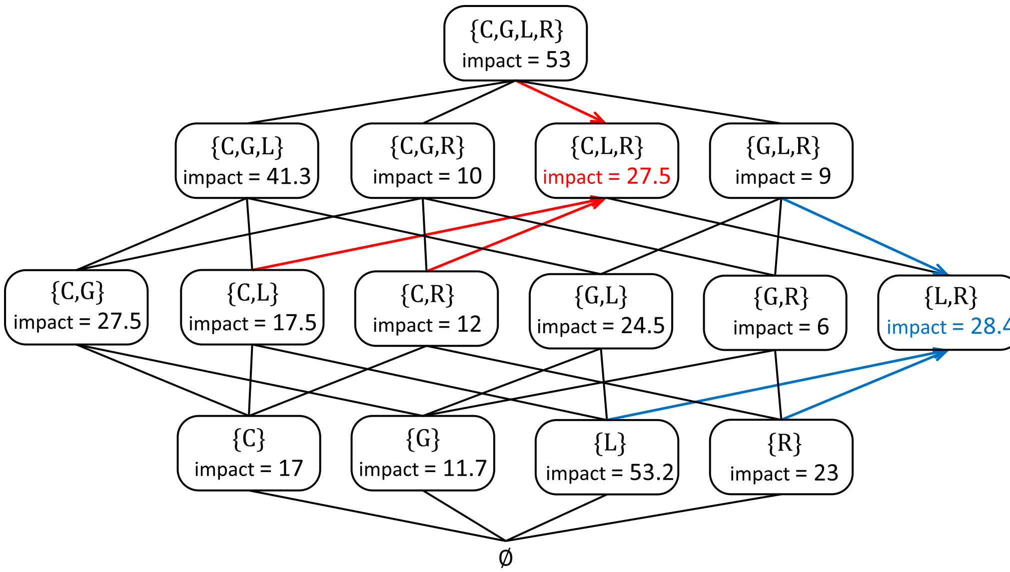

Fig. 5. The average 5-year impact of each coalition of the scientists: Diego (C)alvanese, Maurizio (L)enzerini, Giuseppe De (G)iacomo, and Riccardo (R)osati. The authors do not yet have the following two coalitions: $\{C, L, R\}$ and $\{L, R\}$. The values of these missing coalitions were assumed to be equal to the average of all coalitions with known values that are connected to them in the subset lattice (the arrows highlight the values that were averaged to obtain the values of the missing coalitions).

Table 5. Given the Game Defined in Figure 5, the Table Compares the Added-Value Index, χ_v^N, Against the Other Synergy Measures, Namely: ψ^{it}—the Measure from Information Theory; ψ^{Hd}—the Measure Based on the Harsanyi Dividends; ψ^{ii}—the Measure Based on the Interaction Index, and ψ^{sop}—the Measure Whereby Synergy Is the Surplus Over the Optimal Partition

S	$v(S)$	$\psi^{it}(S)$ Section 7.1	$\psi^{Hd}(S)$ Section 7.2	$\psi^{ii}(S)$ Section 7.3	$\psi^{sop}(S)$ Section 7.4	$\chi_v^N(S)$ Section 4
$\{C\}$	17	0	17	14.3	17	10.16
$\{G\}$	11.7	0	11.7	6.57	11.7	9.33
$\{L\}$	53.17	0	53.17	28.3	53.17	23.09
$\{R\}$	23	0	23	3.8	23	17.12
$\{C, G\}$	27.5	−1.2	−1.2	25.82	−1.2	18.29
$\{C, L\}$	17.5	−52.67	−52.67	−2.38	−52.67	−19.45
$\{C, R\}$	12	−28	−28	3.57	−28	−0.72
$\{G, L\}$	24.5	−40.37	−40.37	−2.48	−40.37	−7.98
$\{G, R\}$	6	−28.7	−28.7	−9.53	−28.7	−2.25
$\{L, R\}$	28.4	−47.77	−47.77	−5.33	−47.77	−7.59
$\{C, G, L\}$	41.3	−40.57	53.67	41.8	−40.57	1.98
$\{C, G, R\}$	10	−41.7	16.2	4.3	−41.7	−5.09
$\{C, L, R\}$	27.5	−65.67	62.77	50.9	−65.67	−15.33
$\{G, L, R\}$	9	−78.87	37.97	26.1	−78.87	−29.36
$\{C, G, L, R\}$	53	−51.87	−23.77	−23.8	−51.87	7.8

Synergies are highlighted in blue, while antergies are highlighted in red.

of the Average-Impact index are as follows: 6.84 for C, 2.37 for G, 30.08 for L, and 5.88 for R; these are the expected contributions of each scientist to the citations of a paper. We observe that the citations attained by each of them when working alone exceed expectations, hence the positive added value. Notice how expectations are based on the performance of each scientist in various groups and not just in single-authored papers.

— *Pairs*: the Added-Value index indicates that, except for $\{C, G\}$, the pairs fall short of expectations. This mostly agrees with the other synergy measures. However, we observe that the antergies according to the Added-Value index are significantly smaller than those according to ψ^{it}, ψ^{Hd}, and ψ^{sop}. This is because, for pairs, the benchmark for the latter three indices are the single-authored papers, the citations of which are comparatively very high. In contrast, the benchmark for our measure are the values obtained from the Average-Impact index, which is constructed from the entire characteristic function (again, this resembles the situation in *Game 3*, with positive χ_v^N for singletons while negative for bigger coalitions). These differences (between our benchmark and that of the aforementioned measures) are perhaps most evident in $\{C, G\}$, which, according to ψ^{it}, ψ^{Hd}, and ψ^{sop}, has negative synergy; while; according to our Added-Value index, has positive synergy. Finally, commenting on ψ^{ii}, we find that the values of synergy according to this measure are, for most pairs, not too far from those of the Added-Value index for $\{C, G\}$. This is because, although ψ^{ii} compares the performance of $\{C, G\}$ to that of $\{C\}$ and $\{G\}$ (just like ψ^{it}, ψ^{Hd}, and ψ^{sop}), the comparison is done differently, taking into consideration all groups to which $\{C\}$, $\{G\}$, and $\{C, G\}$ belong to.

— *Triplets*: The results resemble those for pairs. In particular, just like ψ^{it} and ψ^{sop}, the Added-Value index recognizes that these coalitions underperform, with the exception of $\{C, G, L\}$. However, due to the different benchmark used by our index, the antergies are less pronounced. As for the coalition $\{C, G, L\}$, while our measure assigns to it a positive synergy (just like ψ^{Hd} and ψ^{ii}), the reasons behind the positive synergy are different. In our case, the performance of $\{C, G, L\}$ slightly exceeded expectations (taking into consideration the expected contribution of a member to various groups), while for ψ^{Hd} and ψ^{ii}, the high positive synergy is determined based on the antergies in the pairs.

— *The grand coalition*: for this coalition the Added-Value index reports positive synergy while all the other measures report negative synergy. This is because, by focusing solely on the exceptionally high performance of singletons (just like ψ^{it} and ψ^{sop} do), one would conclude that the synergy in the grand coalition is negative. Alternatively, if the *positive* synergy in triples is inferred from the *negative* synergy in pairs, which is in turn inferred from the *positive* synergy in singletons, then one would arrive at the conclusion that the synergy in the grand coalition is *negative* (just like ψ^{Hd} and ψ^{ii} do). In contrast, if expectations are based on the entire game, including the exceptional groups as well as the underperforming ones, we arrive at the conclusion that the performance of the grand coalition actually exceeds expectations, although just by a little bit.

In summary, with the above examples, we have demonstrated how our index quantifies the added value in a coalition based on the interactions not only *inside* the coalition but also *outside* of it, thereby revealing information about how the members typically perform in different groups of the population. Importantly, we do not claim that the Added-Value index is superior to the other measures of synergy, but rather that it captures certain aspects of synergy that other measures seem to miss. Likewise, as the example demonstrates, there are certain other aspects of synergy that are better captured by the other measures. As such, our measure does not dominate the other synergy measures, but rather complements them.

7.6 Non-Combinatorial Studies of Synergy

Synergy has been studied in a wide variety of other disciplines, including computer science, physics, biology, medicine, and social sciences. These studies are predominantly focused on understanding the nature of synergy in very specific applications or domains. In other words, they propose various application-specific definitions and measures of synergy, most often focusing on synergy between just two entities. Thus, unlike our measure of added value, or the other measures of synergy discussed in previous subsections, these studies do not consider the combinatorial aspects that typically emerge when an arbitrary number of entities is involved. Below, we briefly discuss some of these studies.

Synergy is a recurring theme in computer science, especially in the domain of multi-agent systems, the fundamental idea of which is to have software agents work together to improve their efficiency. For instance, by identifying any overlapping effects among the agents' plans, some agents are able to leave some tasks to others (Cox and Durfee 2003). The synergy here represents the reduced total cost of task execution, and the definition of synergy that fits this application is the one from Equation (67). A specific model of synergy in multi-agent systems was proposed by Liemhetcharat and Veloso (2012). The idea is based on the *synergy graph*—a weighted undirected graph that defines pairwise compatibility between agent types. Given that the compatibility is transitive, the synergy between any two agents is a function of the shortest distance between the nodes representing the agents' types in the synergy graph.

The concept of synergy emerges also in various fields of physics, from mechanics, through thermodynamics to quantum physics. For instance, synergistic effects from the simultaneous use of two different types of lubricants are quantified in tribology (Ouyang et al. 2004).

Synergy also plays a prominent role in medicine and biology. It is particularly relevant when studying the interactions among drugs, as the administration of combinations of drugs often produces unexpected responses. Such an effect is sought after in some treatments and is feared in others, depending on the context. The literature on drug synergies is vast and predominantly occupied with quantifying the supra-additive effects from the addition of different dose-effect curves; see Geary (2013) and Breitinger (2012) for more details. We also mention epidemiology, where a synergy index was proposed to measure the relationship of joint effects from any two exposures compared to the effects of individual exposures (Lee 2013).

In biology, the interactions among different factors take place at all levels of the organization of living organisms, from genetic, through organismic, to societal. For instance, Mesterton-Gibbons and Sherratt (2007) propose an analytical model of coalition formation among the representatives of a species that seek alliances if they deem themselves too weak to secure the resources alone. As most other papers in this body of research, the main focus of Mesterton-Gibbons and Sherratt (2007) is to identify whether certain coalitions of two or three players would yield positive or negative added value, where synergy is understood as the value added over the best partition (see Equation (67)). A well-known in-depth study of the role of synergy in biology, and evolution in particular, can be found in the study by Corning (2005).

The issue of synergy was also considered in various other disciplines such as political economy, development economics, and psychology. In economics, the term synergy typically refers to value added from the merging of two or more companies; such merges create opportunities unavailable if the companies operate independently (Damodaran 1994). As already mentioned, this understanding of synergy corresponds to the value-added-over-the-best-partition definition from Section 7.4. However, creating the value added can be considered as a prerequisite of nearly any productive activity, but it is not usually regarded as synergy. A profit from producing a good or supplying a service is viewed simply as a return on capital invested in the production process. In other words, some (famously, Friedman (1975)) argue that there is no such thing as a "free lunch" in economics. Among the famous works that argue against such a pessimistic approach are those of Henry George who developed political economic theory whereby the introduction of proper tax incentives (the land-value tax, in particular) makes it possible to achieve synergy in the economic activity, where the whole becomes greater than the sum of its parts (Johnson 1910). While George's ideas attracted some fierce criticism, they have many followers, with the land-value tax being implemented in a number of countries and regions.

Achieving synergy between various development policies is often viewed as a *sine qua non* condition of sustainable development. For instance, the EU 2020 strategy postulates a transition to a renewed growth model that fosters the synergies between the economic, social, and environmental policy dimensions (Notre-europe 2012).

Similarly, Leydesdorff and Park (2014) study synergies arising from cooperation between universities, industry, and the government. A rigorous mathematical treatment of how to measure synergy between different dimensions of sustainable development, such as the gross domestic product (GDP), CO_2 emissions, employment, poverty, and outcome distribution, can be found in the work by Luukkanen et al. (2012). Furthermore, we note that the study of synergy in human cooperation is an important research theme in psychology. In this context, the key question is whether the interaction among group members leads to a better performance than would otherwise be achieved by combining the separate efforts of the same people working independently (Larson 2010). Finally, for completion, we mention the extensive quasi-philosophical study of synergy by Buckminster Fuller, who took a holistic, multi-faceted approach towards this phenomenon (Edmondson 1987).

8 CONCLUSIONS

Although added value from group formation arises in numerous diverse settings, a satisfactory mathematical understanding of this phenomenon has remained elusive. This article contributes to the theoretical foundations of measuring added value. In particular, we argued that the quality of the group outcome does not necessarily reflect the added value therein. We also argued that, when added value is measured as the difference between the outcome of the group and the sum or union of the outcomes produced by members working separately, such a measure fails to account for any group-related attributes of the individuals involved. To tackle these issues, we first proposed the *Average-Impact index*—a new solution concept designed to quantify the average impact of each individual in the population, taking into consideration his or her performance in various groups. We also identified a set of properties, or *axioms*, that are all satisfied by the Average-Impact index, and proved that these axioms cannot all be satisfied by any other solution concept. We then used the Average-Impact index as a building block to construct the *Added-Value index*—a new measure of added value. With this measure, the added value is interpreted as a deviation from *expectation* (in terms of the group outcome), whereby each member is expected to perform the way he or she *typically performs in a group*, rather than the way he or she performs *when working alone*. We proposed two alternative theoretical foundations of our measure of our Added-Value index, proving that it is the unique measure of added value that satisfies certain properties, or axioms. In so doing, we provided an informed way to decide when to use this measure. Specifically, in settings where the axioms happen to be meaningful and desirable, the measure would be a reasonable choice, whereas in settings where the axioms seem to be irrelevant, other measures should be considered. Finally, we discussed how our index stands in relation to other concepts of synergy in the literature.

Four directions of future work seem particularly appealing. First, it would be interesting to extend our measure to settings with *externalities*, i.e., influences across co-existing coalitions. Although externalities emerge in a variety of economic situations and attract considerable attention in the literature (de Clippel and Serrano 2008), to the best of our knowledge, no concept of added value in games with externalities has been proposed to date. Second, it would be interesting to study the *computational aspects* of our measure by developing efficient algorithms and concise representations to facilitate polynomial-time computation. Third, it would be interesting to explore the possibility of replacing our measure of average impact with another measure based on past observations, i.e., based on a given "test" set of prior group outcomes. Such a measure would easily account for situations whereby new observations are made and possibly could account for situations whereby new players are added to the population. Finally, it would be interesting to develop a measure of added value that can be applied in settings with uncertainty.

A APPENDIX

A.1 Notation Summary

$N = \{1, \ldots, n\}$	The set of individuals, or "*players*." The notation N also denotes the *grand coalition*, i.e., the coalition consisting of every player in the game.
n	The number of individuals or players, i.e., $\|N\|$.
C or S or T	A *coalition*, i.e., a subset of N.
v	The characteristic function, i.e., the function specifying the *value* of every coalition.
(N, v)	The *characteristic function game*.
$\mathscr{G}(N)$	The set of all possible characteristic function games defined over N.
$MC_i^C(N, v)$	The *marginal contribution* of a player $i \in N$ to a coalition $C \subseteq N$; see Definition 2.6.
$\phi_i(N, v)$	The *Shapley value* of player i in game (N, v); see Definition 2.7.
$\overline{\phi}_i(N, v)$	The *average Shapley value* of player i in game (N, v); see Definition 3.1.
Θ	The set of all possible measures of *average impact*.
$\theta_v^N(i)$	The *average impact* of player i in game (N, v) according to an arbitrary measure of average impact, $\theta \in \Theta$.
$\alpha_v^N(i)$	The *average impact* of player i in game (N, v) according to our measure of average impact, i.e., the *Average-Impact index*, defined as: $\alpha_v^N(i) = \overline{\phi}_i(N, v)$; see Definition 3.1.
Ψ	The set of all possible measures of *value added*.
$\psi_v^N(C)$	The *added value* in coalition C in game (N, v) according to an arbitrary measure of added value, $\psi \in \Psi$.
$\chi_v^N(C)$	The *added value* in coalition C in game (N, v) according to our measure of added value, i.e., the *Added-Value index*; see Definition 4.2.
$\psi^{it}(C)$	The synergy in coalition C according to *information theory*; see Section 7.1.
$\psi^{Hd}(C)$	The synergy in coalition C according to the *Harsanyi dividends*; see Section 7.2.
$\psi^{ii}(C)$	The synergy in coalition C according to the *interaction index*; see Section 7.3.
$\psi^{sop}(C)$	The synergy in coalition C measured as the *surplus over the optimal partition* of C; see Section 7.4.

A.2 The List of the Publications for the Example from Google Scholar

Table 6. A List of All the Papers Published Before 2014 by Various Subsets of the Following Four Scientists: Diego Calvanese, Maurizio Lenzerini, Giuseppe De Giacomo, and Riccardo Rosati; the Authors in Each Subset Are Represented by the First Letter of Their Last Name

Subset	Impact	Year	Publication title
$\{C\}$	20	1996	Finite model reasoning in description logics
	18	1996	Reasoning with inclusion axioms in description logics: Algorithms and complexity
	13	1996	Unrestricted and finite model reasoning in class-based representation formalisms. PhD thesis
$\{G\}$	43	1997	Decidability of class-based knowledge representation formalisms
	15	1996	Eliminating converse from converse PDL
	2	1996	Intensional query answering by partial evaluation
	4	1993	Intensional query answering: an application of partial evaluation
	3	1998	Cognitive robotics
	3	1993	Reconciling different semantics for concept definition
$\{L\}$	760	2002	Data integration: A theoretical perspective
	57	2011	Ontology-based data management
	22	2001	Data integration is harder than you thought
	9	1987	Covering and disjointness constraints in type networks
	20	2004	Principles of P2P data integration.
	5	1999	Description logics and their relationships with databases
	6	1990	Class hierarchies and their complexity
	5	1985	SERM: Semantic entity-relationship model
	8	2003	Information integration
	1	1991	Type data bases with incomplete information
	5	2005	Logical foundations for data integration
	2	2001	Data integration needs reasoning
	2	2004	Principles of peer-to-peer data integration
	1	2006	Inconsistency tolerance in P2P data integration
	0	2005	Logiche descrittive per lintegrazione di dati e servizi
	0	1991	Careful closure of inheritance networks
	1	1987	Formal treatment of incomplete information in type data bases
$\{R\}$	170	2006	DL+log: Tight integration of description logics and disjunctive datalog
	140	2005	On the decidability and complexity of integrating ontologies and rules
	68	2007	On conjunctive query answering in EL
	13	1999	Towards expressive KR systems integrating datalog and description logics: Preliminary report
	47	2005	Semantic and computational advantages of the safe integration of ontologies and rules
	28	2007	The limits of querying ontologies

(Continued)

Table 6. Continued

Subset	Impact	Year	Publication title
	47	2011	On the complexity of dealing with inconsistency in description logic ontologies
	25	2006	On the decidability and finite controllability of query processing in databases with incomplete information
	44	2012	Prexto: Query rewriting under extensional constraints in DL lite
	32	2006	Integrating ontologies and rules: semantic and computational issues
	21	2008	Finite model reasoning in DL-Lite
	29	2011	On the finite controllability of conjunctive query answering in databases under open-world assumption
	15	2008	On combining description logic ontologies and nonrecursive datalog rules
	14	1997	Reasoning with minimal belief and negation as failure: Algorithms and complexity
	14	1999	Reasoning about minimal belief and negation as failure
	5	1999	Model checking for nonmonotonic logics: Algorithms and complexity
	4	2000	On the decidability and complexity of reasoning about only knowing
	1	1998	Autoepistemic description logics
	4	1997	Embedding minimal knowledge into autoepistemic logic
	7	2012	Query rewriting under extensional constraints in DL-Lite
	6	2006	The limits and possibilities of combining description logics and datalog
	3	2001	A sound and complete tableau calculus for reasoning about only knowing and knowing at most
	3	1999	Towards first-order nonmonotonic reasoning
	1	1997	Complexity of only knowing: The prepositional case
	2	2003	Minimal belief and negation as failure in multi-agent systems
	3	2000	Tableau calculus for only knowing and knowing at most
	4	1999	Reasoning about minimal knowledge in nonmonotonic modal logics
	1	1996	On the semantics of epistemic description logics
	3	1998	Reducing query answering to satisfiability in nonmonotonic logics
	3	1997	Minimal knowledge states in nonmonotonic modal logics
	2	1998	Expressiveness vs. complexity in nonmonotonic knowledge bases: Propositional case.
	1	2009	Effective ontology-based data integration
	0	1998	Embedding negation as failure into minimal knowledge
$\{C, G\}$	67	2003	Expressive description logics
	37	2005	Data integration: A logic-based perspective
	4	2003	Description logics for conceptual data modeling in UML
	2	2009	Ontology-based data integration

(Continued)

Table 6. Continued

Subset	Impact	Year	Publication title
$\{C, L\}$	10	1994	On the interaction between ISA and cardinality constraints
	25	1994	Making object-oriented schemas more expressive
$\{C, R\}$	12	2003	Answering recursive queries under keys and foreign keys is undecidable
$\{G, L\}$	66	1996	TBox and ABox reasoning in expressive description logics
	59	1994	Boosting the correspondence between description logics and propositional dynamic logics
	29	1995	PDL-based framework for reasoning about actions
	28	1994	Concept language with number restrictions and fixpoints, and its relationship with mu-calculus
	27	1997	A uniform framework for concept definitions in description logics
	15	1994	Description logics with inverse roles, functional restrictions, and n-ary relations
	9	1995	Enhanced propositional dynamic logic for reasoning about concurrent actions
	6	1995	Making CAT S out of kittens: Description logics with aggregates
	4	1994	Converse, local determinism, and graded nondeterminism in propositional dynamic logics
	2	1994	On the correspondence between description logics and logics of programs
$\{G, R\}$	6	1999	Minimal knowledge approach to reasoning about actions and sensing
$\{C, G, L\}$	100	1998	On the decidability of query containment under constraints
	170	2002	A framework for ontology integration
	94	2000	Answering queries using views over description logics knowledge bases
	56	2001	Ontology of integration and integration of ontologies
	63	2002	Description logics for information integration
	34	1999	Reasoning in expressive description logics with fixpoints based on automata on infinite trees
	50	1999	Representing and reasoning on XML documents: A description logic approach
	30	2001	Identification constraints and functional dependencies in description logics
	55	1995	Structured objects: Modeling and reasoning
	54	2008	Conjunctive query containment and answering under description logic constraints
	42	1998	What can knowledge representation do for semi-structured data?
	15	1997	Conjunctive query containment in description logics with n-ary relations.

(Continued)

Table 6. Continued

Subset	Impact	Year	Publication title
	25	2002	Description logics: Foundations for class-based knowledge representation
	12	1999	Modeling and querying semi-structured data
	10	2002	2ATAs make DLs easy
	10	2000	Keys for free in description logics
	3	1998	Semi-structured data with constraints and incomplete Information
	0	1998	Extending semi-structured data
	2	1997	Representing and reasoning on SGML documents
	1	1996	Representing SGML documents in description logics.
$\{C, G, R\}$	9	1998	A note on encoding inverse roles and functional restrictions in ALC knowledge bases
	11	1999	Data integration and reconciliation in data warehousing: Conceptual modeling and reasoning support
$\{G, L, R\}$	16	2011	Higher-order description logics for domain metamodeling
	6	2009	On higher-order description logics
	5	2008	Towards higher-order DL-Lite (preliminary report)
$\{C, G, L, R\}$	150	2004	Logical foundations of peer-to-peer data integration
	24	2008	View-based query answering over description logic ontologies
	25	2012	View-based query answering in description logics: Semantics and complexity
	13	2011	Actions and programs over description logic knowledge bases: A functional approach

The table also specifies the "impact" of each paper, which we take as the number of citations the paper accumulates five years after its publication, inspired by the work of Sinatra et al. (2016).

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